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# On hierarchical reasoning in combinations of theories

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**Abstract.** In this paper we study theory combinations over non-disjoint signatures in which hierarchical and modular reasoning is possible. We use a notion of locality of a theory extension parameterized by a closure operator on ground terms. We give criteria for recognizing these types of theory extensions. We then show that combinations of extensions of theories which are local in this extended sense have also a locality property and hence allow modular and hierarchical reasoning. We thus obtain parameterized decidability and complexity results for many (combinations of) theories important in verification.

**Key words:** locality, combinations of theories, amalgamation

## 1 Introduction

Many problems in mathematics and computer science can be reduced to proving the satisfiability of conjunctions of literals in a background theory (which can be the extension of a base theory with additional functions – e.g., free, monotone, or recursively defined – or a combination of theories). Considerable work has been dedicated to the task of identifying situations where reasoning in extensions and combinations of theories can be done efficiently and accurately. The most important issues which need to be addressed in this context are: (i) finding possibilities of reducing the search space without losing completeness, and (ii) making modular or hierarchical reasoning possible.

In [10, 18] Givan and McAllester introduced the so-called "local inference systems" (for which validity of ground Horn clauses can be checked in polynomial time). A link between this proof theoretic notion of locality and algebraic arguments used for identifying classes of algebras with a word problem decidable in PTIME [4] was established in [7]. In [8, 22] these results were further extended to so-called *local extensions* of theories. Locality phenomena were also studied in the verification literature, mainly motivated by the necessity of devising methods for efficient reasoning in theories of pointer structures [19] and arrays [3]. In [15] we showed that these results are instances of a general concept of locality of a theory extension – parameterized by a closure operator on ground terms.

Efficient reasoning in combinations of theories is also very important. Methods for checking satisfiability of conjunctions of ground literals in combinations of theories which have disjoint signatures, or only share constants, are well studied. The Nelson/Oppen combination procedure [20] can be applied for combining decision procedures of stably infinite theories over disjoint signatures; various extensions have been established either by relaxing the requirement that the theories to be combined are stably-infinite [27]; or by relaxing the requirement that the theories to be combined have disjoint signatures [1, 26, 9]. These extensions require additional conditions, e.g. generalizations of stable infinity on the component theories, noetherianity of the shared theory.

Since the notion of local extensions we studied [22] imposes no major restrictions on the base theory, it offers interesting, orthogonal criteria for the transfer of decidability in combinations of theories. In this paper we show that efficient reasoning techniques can be provided for combinations of local theory extensions as well. To this end, we present new results on preservation of locality and $\Psi$-locality of theory extensions under theory combinations, extending earlier results in [24] (cf. also [23]). The main results of this paper can be summarized as follows:

- We present semantic characterizations for various notions of locality (possibly parameterized by a closure operator on ground terms).
- We present various results on transfer of locality (and hence also of decidability), some with a model theoretical flavor.
- We identify increasingly complex conditions under which locality is preserved under taking unions of theories. We strengthen some results in [24, 23] by considering the embeddability conditions ($\mathsf{EEmb}_w$) instead of ($\mathsf{Comp}_w$) and by considering combinations of $\Psi_i$-local extensions of a theory.
- We briefly discuss the way these ideas are implemented in H-PILoT.

**Structure of the paper.** The paper is structured as follows. Section 2 contains generalities on theories, local theories, partial algebras, weak validity and embeddability. In Section 3 we present ways of recognizing locality. In Section 4 we give semantical characterizations of locality; these are used in Section 5 to transfer locality results from one theory extension to other theory extensions. Section 6 presents our results on combinations of local theory extensions, and a description of the way we implemented hierarchical reasoning in such combinations.

This report is the extended version of [14].

## 2 Preliminaries

We assume standard definitions from first-order logic. In this paper, (logical) theories are described as sets of sentences (axioms of the theory).

### 2.1 Embeddings and formula preservation

A map $\varphi : \mathcal{A} \to \mathcal{B}$ between two first-order structures *preserves a formula* $F(\bar{x})$, with $\bar{x} = (x_1, \dots, x_n)$, if it holds for every sequence of elements $\bar{a} = (a_1, \dots, a_n)$

of $\mathcal{A}$ that

$$\mathcal{A} \models F(\bar{a}) \Rightarrow \mathcal{B} \models F(\varphi(\bar{a})),$$

where $\varphi(\bar{a})$ is the sequence $(\varphi(a_1), \dots, \varphi(a_n))$. For example, $\varphi$ is a homomorphism if and only if it preserves all atomic formulae; $\varphi$ is an embedding if and only if it preserves all literals (i.e., atomic formulae and negations of atomic formulae) – which implies that all *quantifier-free* formulae are preserved. Equally important is the case when $\varphi$ preserves *all* formulae.

A map $\varphi : \mathcal{A} \to \mathcal{B}$ between two first-order structures is an *elementary embedding* if and only if it preserves all formulae, i.e., for every formula $F(\bar{x})$ with free variables $\bar{x} = (x_1, \dots, x_n)$ and all elements $\bar{a} = (a_1, \dots, a_n)$ from $A$,

$$\mathcal{A} \models F(\bar{a}) \Rightarrow \mathcal{B} \models F(\varphi(\bar{a})).$$

For example, every isomorphism is an elementary embedding. If an elementary embedding $\varphi$ is the inclusion, we say that $\mathcal{A}$ is an *elementary substructure of* $\mathcal{B}$ (notation: $\mathcal{A} \preccurlyeq \mathcal{B}$). Two structures $\mathcal{A}, \mathcal{B}$ are *elementarily equivalent* (notation: $\mathcal{A} \equiv \mathcal{B}$) if they satisfy the same sentences.

Note that if there is an elementary embedding between two structures, then they are elementarily equivalent in particular.

### 2.2 Extensions of theories

Let $\Pi_0{=}(\Sigma_0, \mathsf{Pred})$ be a signature, and $\mathcal{T}_0$ be a "base" theory with signature $\Pi_0$. We consider the following types of extensions of $\mathcal{T}_0$:

- *Extensions with sets of clauses* are extensions $\mathcal{T} := \mathcal{T}_0 \cup \mathcal{K}$ of $\mathcal{T}_0$ with new function symbols $\Sigma$ (called *extension functions*) whose properties are axiomatized using a set $\mathcal{K}$ of (universally closed) clauses in the extended signature $\Pi = (\Sigma_0 \cup \Sigma, \mathsf{Pred})$. We assume that every clause in $\mathcal{K}$ contains function symbols in $\Sigma$.
- *Extensions with augmented clauses* are extensions $\mathcal{T} := \mathcal{T}_0 \cup \mathcal{K}$ with new function symbols $\Sigma$ whose properties are axiomatized using a set $\mathcal{K}$ of formulae of the form $D = \forall \bar{x}\, (\Phi(\bar{x}) \vee C(\bar{x}))$ where $\Phi(\bar{x})$ is an *arbitrary formula* in the base signature $\Pi_0$ and $C(\bar{x})$ is a clause in the extended signature $\Pi$, which contains at least one function symbol of $\Sigma$.
  If for every formula $D \in \mathcal{K}$, $\Phi(\bar{x})$ is universal we speak of *extension by universal augmented clauses*; if $\Phi(\bar{x})$ belongs to a certain class $\mathcal{F}$ of $\Pi_0$-formulae we speak of *extension by $\mathcal{F}$-augmented clauses.*

*Example 1.* The following examples illustrate the notions above:

(i) Let $\mathcal{T}_0$ be the theory of Presburger arithmetic with signature $\Pi_0$. Let $\Sigma = \{f\}$ where $f$ is a new function symbol, and let $\mathcal{K}_f = \{\forall x, y, z\, (y \neq z \to f(x, y) \neq f(x, z))\}$ be an axiomatization for the injectivity of $f$ in its second argument. Then $\mathcal{T}_1 := \mathcal{T}_0 \cup \mathcal{K}_f$ is an extension of $\mathcal{T}_0$ with the set $\mathcal{K}$ of clauses and $f$ is an extension function.

(ii) Let $\mathcal{T}_1$ be the theory defined at (i) and let $\Sigma = \{g\}$, where $g \notin \Sigma_0 \cup \Sigma$ and let $\mathcal{K}_g = \{\forall x, y([\forall z(z \neq y \rightarrow f(x,z) < f(x,y))] \rightarrow g(x) = f(x,y))\}$. Then $\mathcal{T} := \mathcal{T}_1 \cup \mathcal{K}_g$ is an extension of $\mathcal{T}_1$ with a set $\mathcal{K}_g$ of augmented clauses (in fact $\mathcal{F}$-augmented clauses where $\mathcal{F}$ is the $\exists$-fragment of $\mathcal{T}_1$).

Our goal is to address proof tasks of the form $G \models_{\mathcal{T}_0 \cup \mathcal{K}} \perp$, where $G$ is a set of ground clauses with additional (fresh) constants (in a countable set $C$), i.e. in the signature $\Pi^C = (\Sigma_0 \cup \Sigma \cup C, \mathsf{Pred})$.

For the case of extensions $\mathcal{T}_0 \cup \mathcal{K}$ by augmented clauses we also consider the more general task of checking satisfiability problems of the form $\Gamma \models_{\mathcal{T}_0 \cup \mathcal{K}} \perp$, where $\Gamma$ is a conjunction of sentences of the form $\Psi_0 \vee C$, where $C$ is a ground clause in the signature $\Pi^C$, and $\Psi_0$ is a $\Pi_0^C$-sentence.

We also consider combinations of extensions $(\mathcal{T}_0 \cup \mathcal{K}_1)$ and $(\mathcal{T}_0 \cup \mathcal{K}_2)$ of the base theory $\mathcal{T}_0$, where $\mathcal{K}_i$ are sets of (augmented) clauses over $(\Sigma_0 \cup \Sigma_i, \mathsf{Pred})$. Our proof tasks $G$, then, will be in the signature $(\Sigma_0 \cup \Sigma_1 \cup \Sigma_2 \cup C, \mathsf{Pred})$. Using new constants, we can always separate $G$ into a $(\Sigma_0 \cup \Sigma_1 \cup C, \mathsf{Pred})$-part $G_1$, a $(\Sigma_0 \cup \Sigma_2 \cup C, \mathsf{Pred})$-part $G_2$ part, and a $\Pi_0$-part $G_0$.

### 2.3 Locality conditions

Let $\mathcal{T}_0$ be an arbitrary theory with signature $\Pi_0 = (\Sigma_0, \mathsf{Pred})$, where the set of function symbols is $\Sigma_0$. Let $\Pi = (\Sigma_0 \cup \Sigma, \mathsf{Pred}) \supseteq \Pi_0$ be an extension by a non-empty set $\Sigma$ of new function symbols and $\mathcal{K}$ be a set of (implicitly universally closed) clauses in the extended signature. Let $C$ be a fixed countable set of fresh constants. We say that an extension $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}$ of the above form is *local* if it satisfies the following condition

(Loc) For every set $G$ of ground clauses in $\Pi^C$ it holds that
$\mathcal{T}_0 \cup \mathcal{K} \cup G \models \perp$ if and only if $\mathcal{T}_0 \cup \mathcal{K}[G] \cup G \models \perp$

where $\mathcal{K}[G]$ consists of those instances of $\mathcal{K}$ in which the terms starting with *extension functions* are in the set $\mathsf{est}(\mathcal{K}, G)$ of extension ground terms (i.e. terms starting with a function in $\Sigma$) which already occur in $G$ or $\mathcal{K}$.
The notion of local theory extension generalizes the notion of *local theories* [10, 18, 11, 7]. In [15] we generalized the notion of locality by considering operators on ground terms. This allows us to be more flexible w.r.t. the instances needed.

**Definition 2.** *With the notations above, let $T$ be a set of ground terms in the signature $\Pi^C$. We denote by $\mathcal{K}[T]$ the set of all instances of $\mathcal{K}$ in which the terms starting with a function symbol in $\Sigma$ are in $T$. Formally:*

$\mathcal{K}[T] := \{\varphi\sigma \mid \forall \bar{x}. \varphi(\bar{x}) \in \mathcal{K},$ *where (i) if $f \in \Sigma$ and $t = f(t_1, ..., t_n)$ occurs in $\varphi\sigma$ then $t \in T$; (ii) if $x$ is a variable that does not appear below some $\Sigma$-function in $\varphi$ then $\sigma(x) = x\}$.*

**Definition 3.** *Let $\Psi$ be a map associating with every set $T$ of ground terms a set $\Psi(T)$ of ground terms. For any set $G$ of (augmented) ground $\Pi^C$-clauses we write $\mathcal{K}[\Psi_{\mathcal{K}}(G)]$ for $\mathcal{K}[\Psi(\mathsf{est}(\mathcal{K}, G))]$. We define the following versions of locality in which the set of terms used for constructing the instances of the axioms is described using the map $\Psi$:*
*Let $\mathcal{T}_0 \cup \mathcal{K}$ be an extension of $\mathcal{T}_0$ with clauses in $\mathcal{K}$. We define:*

($\mathsf{Loc}^{\Psi}$) *For every set $G$ of ground clauses in $\Pi^C$ it holds that $\mathcal{T}_0 \cup \mathcal{K} \cup G \models \perp$ if and only if $\mathcal{T}_0 \cup \mathcal{K}[\Psi_{\mathcal{K}}(G)] \cup G \models \perp$.*

*Let $\mathcal{T}_0 \cup \mathcal{K}$ be an extension of $\mathcal{T}_0$ with augmented clauses in $\mathcal{K}$. We define:*

($\mathsf{ELoc}^{\Psi}$) *For every set of formulas $\Gamma = \Gamma_0 \cup G$, where $\Gamma_0$ is a $\Pi_0^C$-sentence and $G$ is a set of ground $\Pi^C$-clauses, it holds that $\mathcal{T}_0 \cup \mathcal{K} \cup \Gamma \models \perp$ if and only if $\mathcal{T}_0 \cup \mathcal{K}[\Psi_{\mathcal{K}}(G)] \cup \Gamma \models \perp$.*

*Extensions satisfying condition ($\mathsf{Loc}^{\Psi}$) are called $\Psi$-local; we refer to ($\mathsf{ELoc}^{\Psi}$) as the* extended $\Psi$-locality *condition.*
Finite *locality and extended finite $\Psi$-locality (notation: ($\mathsf{Loc}_f^{\Psi}$)/($\mathsf{ELoc}_f^{\Psi}$)) can also be defined, by specifying that the locality conditions hold for all* finite *sets $G$ of ground clauses.*

*Remark 4.* In [22] and [15] we defined locality, respectively $\Psi$-locality, w.r.t. weak partial models in which the relevant extension terms ($\mathsf{est}(\mathcal{K}, G)$/ $\Psi_{\mathcal{K}}(G)$) are defined. There, a given extension $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}$ was said to be local, if for any set of ground clauses $G$, we have $\mathcal{T}_0 \cup \mathcal{K} \cup G \models \perp$ if and only if $\mathcal{T}_0 \cup \mathcal{K}[G] \cup G$ had no weak partial model in which all terms in $\mathsf{est}(\mathcal{K}, G)$ were defined. It is easy to see that these formulations are equivalent: On the one hand, every total model is also a partial one. On the other hand, given such a weak partial model, we can make its functions total by giving them a default value, say, on points where they are undefined. Since this does not affect the interpretation of any terms appearing in $\mathcal{K}[G] \cup G$, satisfiability is not affected. A similar argument applies to $\Psi$-locality.

*Example 5.* Local theory extensions are $\Psi$-local, where $\Psi$ is the identity operator. The order-local theories introduced in [2] satisfy a $\Psi$-locality condition, where for every set $T$ of ground clauses $\Psi(T) = \{s \mid s \text{ ground term and } s \preceq t \text{ for some } t \in T\}$, where $\prec$ is the order on terms considered in [2].

### 2.4 Hierarchical reasoning in local theory extensions

Let $\mathcal{T}_0 \subseteq \mathcal{T}{=}\mathcal{T}_0{\cup}\mathcal{K}$ be a theory extension satisfying condition (($\mathsf{E}$)$\mathsf{Loc}^{\Psi}$). To check the satisfiability w.r.t. $\mathcal{T}$ of a formula $\Gamma_0 \cup G$, where $\Gamma_0$ is a $\Pi_0^C$-sentence[1] and $G$ is a set of ground $\Pi^C$-clauses, we proceed as follows:

**Step 1:** By locality, $\mathcal{T}{\cup}\Gamma_0{\cup}G \models\perp$ if and only if $\mathcal{T}_0{\cup}\mathcal{K}[\Psi_{\mathcal{K}}(G)]{\cup}\Gamma_0{\cup}G \models\perp$.

[1] In the case of condition ($\mathsf{Loc}^{\Psi}$), $\Gamma_0 = \top$.

**Step 2: Purification.** We purify $\mathcal{K}[\Psi_\mathcal{K}(G)] \cup G$ by introducing, in a bottom-up manner, new constants $c_t$ for subterms $t = f(g_1, \dots, g_n)$ with $f \in \Sigma$, $g_i$ ground $(\Sigma_0 \cup C)$-terms, together with their definitions $c_t \approx t$. The set of formulae thus obtained has the form $\mathcal{K}_0 \cup G_0 \cup \Gamma_0 \cup D$, where $D$ consists of definitions of the form $f(g_1, \dots, g_n) \approx c$, where $f \in \Sigma$, $c$ is a constant, $g_1, \dots, g_n$ are ground $\Pi_0^C$-terms, and $\mathcal{K}_0, G_0, \Gamma_0$ are $\Pi_0^C$-formulae.

**Step 3: Reduction to testing satisfiability in $\mathcal{T}_0$.** We reduce the problem to testing satisfiability in $\mathcal{T}_0$ by replacing $D$ with the following set of clauses:

$$\mathsf{Con}_0 = \{\bigwedge_{i=1}^{n} c_i \approx d_i \rightarrow c = d \mid f(c_1, \dots, c_n) \approx c, f(d_1, \dots, d_n) \approx d \in D\}.$$

This yields a sound and complete hierarchical reduction to a satisfiability problem in the base theory $\mathcal{T}_0$:

**Theorem 6 ([15]).** *Let $\mathcal{K}$ and $\Gamma_0 \wedge G$ be as specified above. Assume that $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}$ satisfies condition $((\mathsf{E})\mathsf{Loc}^{\Psi})$. Let $\mathcal{K}_0 \cup G_0 \cup \Gamma_0 \cup \mathsf{Con}_0$ be obtained from $\mathcal{K}[\Psi_\mathcal{K}(G)] \cup \Gamma_0 \cup G$ by purification (cf. Step 2). The following are equivalent:*

*(1)* $\mathcal{T}_0 \cup \mathcal{K} \cup \Gamma_0 \cup G \models \perp$ .
*(2)* $\mathcal{T}_0 \cup \mathcal{K}_0 \cup G_0 \cup \Gamma_0 \cup \mathsf{Con}_0 \models \perp$.

*Thus, satisfiability of goals $\Gamma_0 \cup G$ as above w.r.t. $\mathcal{T}$ is decidable provided $\mathcal{K}[\Psi_\mathcal{K}(G)]$ is finite and $\mathcal{K}_0 \cup G_0 \cup \Gamma_0 \cup \mathsf{Con}_0$ belongs to a decidable fragment of $\mathcal{T}_0$.*

**Implementation.** This method is implemented in the program H-PILoT (Hierarchical Proving by Instantiation in Local Theory Extensions) ([13]). H-PILoT carries out a hierarchical reduction to $\mathcal{T}_0$ step-by-step if the user specifies different levels for the extension functions in a chain of theory extensions. Standard SMT provers or specialized provers can be used for testing the satisfiability of the formulas obtained after the reduction. If the result of the reduction is a satisfiable problem, H-PILoT is able to *generate a model*. $\Psi$-locality is handled for the *array property fragment* and a fragment of the theory of pointers [3, 19, 15], which are fully integrated into H-PILoT. In this paper we establish ways of recognizing ($\Psi$-)locality for wider classes of theories.

### 2.5 Partial structures

Local extensions can be recognized by showing that certain partial models embed into total ones. We introduce the main definitions here.

Let $\Pi = (\Sigma, \mathsf{Pred})$ be a first-order signature with set of function symbols $\Sigma$ and set of predicate symbols $\mathsf{Pred}$. A *partial $\Pi$-structure* is a structure $\mathcal{A} = (A, \{f_\mathcal{A}\}_{f \in \Sigma}, \{P_\mathcal{A}\}_{P \in \mathsf{Pred}})$, where $A$ is a non-empty set, for every $f \in \Sigma$ with arity $n$, $f_\mathcal{A}$ is a partial function from $A^n$ to $A$, and for every $P \in \mathsf{Pred}$, with arity $n$, $P_\mathcal{A} \subseteq A^n$. We consider constants (0-ary functions) to be always defined. $\mathcal{A}$ is called a *total structure* if the functions $f_\mathcal{A}$ are all total. Given a (total or partial) $\Pi$-structure $\mathcal{A}$ and $\Pi_0 \subseteq \Pi$ we denote the reduct of $\mathcal{A}$ to $\Pi_0$ by $\mathcal{A}|_{\Pi_0}$.

Evaluating a term $t$ with variables $X$ w.r.t. an assignment $\beta{:}X{\to}A$ for its variables in a partial structure $\mathcal{A}$ is done as for total algebras, except that the evaluation is undefined if $t = f(t_1, \dots, t_n)$ and at least one of $\beta(t_i)$ is undefined, or else $(\beta(t_1), \dots, \beta(t_n))$ is not in the domain of $f_{\mathcal{A}}$.

Recall that for total $\Pi$-structures $\mathcal{A}$ and $\mathcal{B}$, $\varphi : \mathcal{A} \to \mathcal{B}$ is an embedding if and only if it is an injective homomorphism and has the property that for every $P \in \mathsf{Pred}$ with arity $n$ and all $(a_1, \dots, a_n) \in A^n$, $(a_1, \dots, a_n) \in P_{\mathcal{A}}$ if and only if $(\varphi(a_1), \dots, \varphi(a_n)) \in P_{\mathcal{B}}$. A similar notion can be defined for partial structures.

**Definition 7 (Weak $\Pi$-embedding).** *A* weak $\Pi$-embedding *between partial $\Pi$-structures $\mathcal{A} = (A, \{f_{\mathcal{A}}\}_{f\in\Sigma}, \{P_{\mathcal{A}}\}_{P\in\mathsf{Pred}})$ and $\mathcal{B} = (B, \{f_{\mathcal{B}}\}_{f\in\Sigma}, \{P_{\mathcal{B}}\}_{P\in\mathsf{Pred}})$ is a total map $\varphi : A \to B$ such that*

*(1) whenever $f_{\mathcal{A}}(a_1, \dots, a_n)$ is defined (in $\mathcal{A}$), then $f_{\mathcal{B}}(\varphi(a_1), \dots, \varphi(a_n))$ is defined (in $\mathcal{B}$) and $\varphi(f_{\mathcal{A}}(a_1, \dots, a_n)) = f_{\mathcal{B}}(\varphi(a_1), \dots, \varphi(a_n))$, for all $f \in \Sigma$;*
*(2) for every $P \in \mathsf{Pred}$ with arity $n$ and every $a_1, \dots, a_n \in \mathcal{A}$, $(a_1, \dots, a_n) \in P_{\mathcal{A}}$ if and only if $(\varphi(a_1), \dots, \varphi(a_n)) \in P_{\mathcal{B}}$.*

**Definition 8 (Weak validity).** *Let $\mathcal{A}$ be a partial $\Pi$-algebra and $\beta : X{\to}A$ a valuation for its variables. We define weak validity w.r.t. $(\mathcal{A}, \beta)$ as follows:*

*(1) $(\mathcal{A}, \beta) \models_w t \approx u$ if (i) both $\beta(t)$ and $\beta(u)$ are defined and equal; or (ii) at least one of the terms $\beta(t)$, $\beta(u)$ is undefined.*
*(2) $(\mathcal{A}, \beta) \models_w t \not\approx u$ if (i) both $\beta(t)$ and $\beta(u)$ are defined but different; or (ii) at least one of the terms $\beta(t)$, $\beta(u)$ is undefined.*
*(3) $(\mathcal{A}, \beta) \models_w P(t_1, \dots, t_n)$ if (i) $\beta(t_1), \dots, \beta(t_n)$ are all defined and $(\beta(t_1), \dots, \beta(t_n)) \in P_{\mathcal{A}}$; or (ii) at least one $\beta(t_i)$, $1 \leq i \leq n$, is undefined.*
*(4) $(\mathcal{A}, \beta) \models_w \neg P(t_1, \dots, t_n)$ if (i) $\beta(t_1), \dots, \beta(t_n)$ are all defined and $(\beta(t_1), \dots, \beta(t_n)) \notin P_{\mathcal{A}}$; or (ii) at least one $\beta(t_i)$, $1 \leq i \leq n$, is undefined.*

$(\mathcal{A}, \beta)$ weakly satisfies a clause $C$ *(notation: $(\mathcal{A}, \beta) \models_w C$) if it satisfies at least one literal in $C$. $\mathcal{A}$ is a* weak partial model *of a set of clauses $\mathcal{K}$ if $(\mathcal{A}, \beta) \models_w C$ for every valuation $\beta$ and every clause $C$ in $\mathcal{K}$.*

If $\mathcal{T} = \mathcal{T}_0 \cup \mathcal{K}$ is an extension of a $\Pi_0$-theory $\mathcal{T}_0$ with new function symbols in $\Sigma$ and (augmented) clauses $\mathcal{K}$, we denote by $\mathsf{PMod}_w(\Sigma, \mathcal{T})$ the set of weak partial models of $\mathcal{T}$ whose $\Sigma_0$-functions are total.

### 2.6 Expanding the language

Let $\Pi$ be a signature and let

$$\mathcal{A} = (A, \{f_A\}_{f\in\Sigma}, \{P_A\}_{P\in\mathsf{Pred}})$$

be a partial or total $\Pi$-structure. We consider the extension $\Pi^C$ of the signature $\Pi$ with fresh constants $C = \{c_i \mid i \in I\}$. We denote by $(\mathcal{A}, \bar{b})$ the expansion of $\mathcal{A}$ to a structure for the extended language $\Pi^C$, where $c_i$ is interpreted as $b_i$, for all $i$.

Of particular interest is the case where we have a (fresh) constant $a$ for each element $a$ in the universe $A$ of a structure $\mathcal{A}$. We denote the $\Pi^A$-structure where each new constant $a$ is interpreted by the element $a$ by $(\mathcal{A}, A)$. (As is customary, our notation will not distinguish between the constant $a$ and the element $a$ it names.)

This type of language expansion is an interesting case because of the importance of *Robinson diagrams.* The set of ground $\Pi^A$-literals true in $(\mathcal{A}, A)$ is called the *diagram* of $\mathcal{A}$, the set of $\Pi^A$-sentences true in $(\mathcal{A}, A)$ is called the *elementary diagram* of $\mathcal{A}$. $\mathcal{A}$ is embedded into another structure $\mathcal{B}$ (via some $\varphi$) if and only if $(\mathcal{B}, \varphi\bar{a})_{a\in A}$ is a model of the diagram of $\mathcal{A}$, $\mathcal{A}$ is elementarily embedded into another structure $\mathcal{B}$ (via $\varphi$) if and only if $(\mathcal{B}, \varphi\bar{a})_{a\in A}$ is a model of the elementary diagram of $\mathcal{A}$. Also note that a map $\varphi : \mathcal{A} \to \mathcal{B}$ between two $\Pi$-structures is an elementary embedding if and only if $(\mathcal{A}, \bar{a}) \equiv (\mathcal{B}, \varphi\bar{a})$.

## 3 Recognizing $\Psi$-local theory extensions

In [22] we proved that if all weak partial models of an extension $\mathcal{T}_0 \cup \mathcal{K}$ of a base theory $\mathcal{T}_0$ with total base functions can be embedded into a total model of the extension, then the extension is local. In [15] we lifted these results to $\Psi$-locality. We recall these results and then extend them to obtain semantical *characterizations* of various types of $\Psi$-locality.

In what follows, let $\mathcal{T}_0$ be a $\Pi_0$-theory, and $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K} = \mathcal{T}$ a theory extension with functions in $\Sigma$ and (augmented) clauses $\mathcal{K}$ and let $\Psi$ be as in Definition 3.

**Definition 9.** *Let $\mathcal{A} = (A, \{f_\mathcal{A}\}_{f\in\Sigma_0\cup\Sigma}, \{P_\mathcal{A}\}_{P\in\mathsf{Pred}})$ be a partial $\Pi^C$-structure with total $\Sigma_0$-functions. We denote by $\Pi^A$ the extension of the signature $\Pi$ with constants from $A$. We denote by $\mathcal{D}(\mathcal{A})$ the following set of ground $\Pi^A$-terms*

$$\mathcal{D}(\mathcal{A}) := \{f(a_1, ..., a_n) \mid f \in \Sigma, a_i \in A, i = 1, \dots, n, f_\mathcal{A}(a_1, ..., a_n) \textit{ is defined } \}.$$

**Notation.** We denote by $\mathsf{PMod}_w^\Psi(\Sigma, \mathcal{T})$ the class of all weak partial models $\mathcal{A}$ of $\mathcal{T} = \mathcal{T}_0 \cup \mathcal{K}$ in which the $\Sigma$-functions are partial and all other functions are total and such that all terms in $\Psi(\mathsf{est}(\mathcal{K}, \mathcal{D}(\mathcal{A}))$ are defined (in the extended structure $(\mathcal{A}, A)$ with constants from $A$).

We consider the following embeddability properties of partial algebras.

($\mathsf{Emb}_w^\Psi$) Every $\mathcal{A} \in \mathsf{PMod}_w^\Psi(\Sigma, \mathcal{T})$ weakly embeds into a total model of $\mathcal{T}$.

($\mathsf{Comp}_w^\Psi$) Every $\mathcal{A} \in \mathsf{PMod}_w^\Psi(\Sigma, \mathcal{T})$ weakly embeds into a total model $\mathcal{B}$ of $\mathcal{T}$ such that $\mathcal{A}|_{\Pi_0}$ and $\mathcal{B}|_{\Pi_0}$ are isomorphic.

Variants ($\mathsf{Comp}_{w,f}^\Psi$) and ($\mathsf{Emb}_{w,f}^\Psi$) can be obtained by requiring embeddability only for extension functions with a finite domain of definition.

When establishing links between locality and embeddability we require that the extension clauses in $\mathcal{K}$ are *flat* (or *quasi-flat*) and *linear* w.r.t. extension functions.

**Definition 10.** *We distinguish between ground and non-ground clauses.*

Non-ground clauses: *An extension clause $D$ is* quasi-flat *when all symbols below a $\Sigma$-function symbol in $D$ are variables or ground $\Pi_0$-terms. $D$ is* flat *when all symbols below a $\Sigma$-function symbol in $D$ are variables. $D$ is* linear *if whenever a variable occurs in two terms of $D$ which start with $\Sigma$-functions, the terms are identical, and no such term contains two occurrences of a variable.*

Ground clauses: *A ground clause $D$ is* flat *if all symbols below a $\Sigma$-function in $D$ are constants. A ground clause $D$ is* linear *if whenever a constant occurs in two terms in $D$ whose root symbol is in $\Sigma$, the two terms are identical, and if no term which starts with a $\Sigma$-function contains two occurrences of the same constant.*

**Definition 11.** *With the above notations, let $\Psi$ be a map associating with $\mathcal{K}$ and a set of $\Pi^C$-ground terms $T$ a set $\Psi_{\mathcal{K}}(T)$ of $\Pi^C$-ground terms. Let $\mathrm{est}(\mathcal{K}, T)$ be the set of* extension subterms of $\mathcal{K}$ and $T$, *i.e. the set of ground terms in $\mathcal{K}$ or $T$ in which a $\Sigma$-function symbol appears. We call $\Psi_{\mathcal{K}}$ a* term closure operator *if the following holds for all sets of ground terms $T, T'$:*

*(1)* $\mathrm{est}(\mathcal{K}, T) \subseteq \Psi_{\mathcal{K}}(T)$,
*(2)* $T \subseteq T' \Rightarrow \Psi_{\mathcal{K}}(T) \subseteq \Psi_{\mathcal{K}}(T')$,
*(3)* $\Psi_{\mathcal{K}}(\Psi_{\mathcal{K}}(T)) \subseteq \Psi_{\mathcal{K}}(T)$,
*(4)* $\Psi_{\mathcal{K}}$ *is invariant under constant renaming, i.e., for any map $h : C \to C$, $\bar{h}(\Psi_{\mathcal{K}}(T)) = \Psi_{\bar{h}\mathcal{K}}(\bar{h}(T))$, where $\bar{h}$ is the canonical extension of $h$ to extension ground terms.*

In [15] we proved that if $\Psi$ is a term closure operator then condition ($\mathsf{Comp}_w^{\Psi}$) implies ($\mathsf{ELoc}^{\Psi}$), provided the extension clauses are flat and linear. An analogous proof shows that ($\mathsf{Emb}_w^{\Psi}$) implies ($\mathsf{Loc}^{\Psi}$).[2] This allowed us to identify many examples of $\Psi$-local theory extensions. In [15] we showed that (i) a decidability result for the array property fragment in [3] is due to the $\Psi$-locality (for a certain $\Psi$) of the corresponding extensions of the many-sorted combination of Presburger arithmetic (for indices) with the given theory of elements, and (ii) a fragment of the theory of pointer structures studied in [19] satisfies a $\Psi$-locality property.

## 4 Semantical characterizations of locality

The aim of this section is to obtain semantical characterizations of the notions of $\Psi$-locality studied here. We first show that $\Psi$-locality implies $\Psi$-embeddability.

**Theorem 12.** *Let $\mathcal{T}_0$ be a $\Pi_0$-theory, $\Pi = (\Sigma_0 \cup \Sigma, \mathsf{Pred})$ and let $\mathcal{K}$ be a set of $\Sigma$-flat clauses in the signature $\Pi$. Let $\Psi_{\mathcal{K}}$ be a term closure operator with the property that for every flat set of ground terms $T$, $\Psi(T)$ is flat.*

[2] It is easy to see that if $\mathcal{T}_0$ is a first-order theory and $\Pi_0$ is finite or countable then in order to prove locality it is sufficient to restrict to countable partial models in the embeddability conditions.

*(1) If $\mathcal{T}_0$ is a first-order theory and the extension $\mathcal{T}_0 \subseteq \mathcal{T}{=}\mathcal{T}_0 \cup \mathcal{K}$ satisfies* $(\mathsf{Loc}^\Psi)$ *then every model in* $\mathsf{PMod}_w^\Psi(\Sigma, \mathcal{T})$ *weakly embeds into a total model of $\mathcal{T}$.*
*(2) If $\mathcal{T}_0 \subseteq \mathcal{T}{=}\mathcal{T}_0 \cup \mathcal{K}$ satisfies* $(\mathsf{ELoc}^\Psi)$ *then every $\mathcal{A} \in \mathsf{PMod}_w^\Psi(\Sigma, \mathcal{T})$ weakly embeds into a total model $\mathcal{B}$ of $\mathcal{T}$ such that restriction of this embedding to the reducts to $\Pi_0$ of $\mathcal{A}$, $\mathcal{B}$ preserves the truth of all first-order $\Pi_0$-formulae.*

*Proof*: (1) Let $\mathcal{A}$ be a partial $\Pi$-algebra with totally defined $\Sigma_0$-functions, which is a model of $\mathcal{T}_0$ and weakly satisfies $\mathcal{K}$, in which all terms in $\Psi_\mathcal{K}(\mathcal{D}(\mathcal{A}))$ are defined. Consider the partial diagram of $\mathcal{A}$, i.e.,

$$\begin{aligned}\Delta(\mathcal{A}) = \{ & f(a_1, \dots a_n) \approx a \mid \text{ if } f_\mathcal{A}(a_1, \dots, a_n) \text{ is defined and equal to } a\} \\ \cup \{ & f(a_1, \dots a_n) \not\approx a \mid \text{ if } f_\mathcal{A}(a_1, \dots, a_n) \text{ is defined and not equal to } a\} \\ \cup \{ & P(a_1, \dots, a_n) \mid P \in \mathsf{Pred} \text{ and } (a_1, \dots, a_n) \in P_\mathcal{A}\} \\ \cup \{ & \neg P(a_1, \dots, a_n) \mid P \in \mathsf{Pred} \text{ and } (a_1, \dots, a_n) \notin P_\mathcal{A}\} \cup \bigwedge_{a \neq a', a, a' \in A} a \not\approx a'.\end{aligned}$$

We prove that $\mathcal{T}_0 \cup \mathcal{K} \cup \Delta(\mathcal{A})$ is consistent, where the elements of $\mathcal{A}$ are regarded as new constants. Assume $\mathcal{T}_0 \cup \mathcal{K} \cup \Delta(\mathcal{A}) \models \perp$. By compactness of first-order logic, $\mathcal{T}_0 \cup \mathcal{K} \cup \Gamma \models \perp$, for some finite subset $\Gamma$ of $\Delta(\mathcal{A})$. We know that $\mathcal{A}$ is a model of $\mathcal{T}_0$. Every term starting with a function symbol in $\Sigma$ contained in the clauses in $\mathcal{K}[\Psi(\mathsf{est}(\mathcal{K}, \Gamma))]$ is either a ground (subterm of a) term occurring in $\Psi(\mathsf{est}(\mathcal{K}, \Gamma))$ (and, hence, a constant $a \in A$, or a term $f(a_1, \dots, a_n) \in \Psi(\mathsf{est}(\mathcal{K}, \Gamma)) \subseteq \Psi_\mathcal{K}(\mathcal{D}(\mathcal{A}))$, i.e. $f_\mathcal{A}(a_1, \dots, a_n)$ is defined), or is a ground subterm in $\mathcal{K}$, i.e. a constant, and hence, defined in $\mathcal{A}$. Therefore, all terms occurring in the clauses in $\mathcal{K}[\Psi_\mathcal{K}(\Gamma)]$ are defined in $\mathcal{A}$, so $\mathcal{A}$ satisfies all these clauses, i.e. $\mathcal{A}$ is a model of $\mathcal{T}_0 \cup \mathcal{K}[\Psi_\mathcal{K}(\Gamma)]$. Since $\Delta(\mathcal{A})$ is obviously true in $\mathcal{A}$ and $\Gamma \subseteq \Delta(\mathcal{A})$, $\mathcal{A}$ is a partial model of $\mathcal{T}_0 \cup \mathcal{K}[\Psi_\mathcal{K}(\Gamma)] \cup \Gamma$, in which all ground terms occurring in $\Psi_\mathcal{K}(\Gamma)$ are defined. This contradicts the fact that $\mathcal{T}$ is a $\Psi$-local extension of $\mathcal{T}_0$. Hence, the assumption that $\mathcal{T}_0 \cup \mathcal{K} \cup \Delta(\mathcal{A}) \models \perp$ was false, so $\mathcal{T}_0 \cup \mathcal{K} \cup \Delta(\mathcal{A})$ has a model $\mathcal{A}'$ in which, therefore, $\mathcal{A}$ weakly embeds.

(2) For proving (2) we can repeat the same line of reasoning replacing $\Delta(\mathcal{A})$ with $\Delta(\mathcal{A}) \cup \Delta^e_{\Pi_0}(\mathcal{A})$, where $\Delta^e_{\Pi_0}(\mathcal{A})$ is the elementary $\Pi_0$-diagram of $\mathcal{A}$, i.e. the set of all first-order $\Pi_0$-sentences true in $(\mathcal{A}, A)$. The rest of the reasoning is similar, taking into account that in the extended locality condition we allow for augmented clauses in $\mathcal{K}$ and for augmented ground clauses in $G$: By compactness there exists a finite augmented ground $\Pi_0^A$-clause $\Gamma \subseteq \Delta(\mathcal{A}) \cup \Delta^e_{\Pi_0}(\mathcal{A})$.

The fact that $\mathcal{A}|_{\Pi_0}$ elementarily embeds into $\mathcal{A}'|_{\Pi_0}$ is a consequence of the fact that $\mathcal{A}'|_{\Pi_0}$ is a model of the elementary diagram of $\mathcal{A}|_{\Pi_0}$. □

The second part of Theorem 12 indicates that for extended $\Psi$-locality we need a notion weaker than completability. Therefore, instead of condition $(\mathsf{Comp}_w^\Psi)$ we now consider embeddings that are elementary w.r.t. the base language.

We consider the following property.

($\mathsf{EEmb}_w$) For every $\mathcal{A} \in \mathsf{PMod}_w(\Sigma, \mathcal{T})$ there is a total model $\mathcal{B}$ of $\mathcal{T}$ and a weak embedding $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ such that the embedding $\varphi : \mathcal{A}|_{\Pi_0} \rightarrow \mathcal{B}|_{\Pi_0}$ is elementary.

The definition generalizes in a natural way to a notion ($\mathsf{EEmb}_w^{\Psi}$), parameterized by a closure term operator $\Psi$ by requiring that the embeddability condition holds for all $\mathcal{A} \in \mathsf{PMod}_w^{\Psi}(\Sigma, \mathcal{T})$ with domain of definition closed under $\Psi$, and to corresponding finite embeddability conditions ($\mathsf{EEmb}_{w,f}^{\Psi}$) analogous to ($\mathsf{Emb}_{w,f}^{\Psi}$). Since every isomorphism is an elementary embedding we have the implications ($\mathsf{Comp}_w$) $\rightarrow$ ($\mathsf{EEmb}_w$) $\rightarrow$ ($\mathsf{Emb}_w$) and ($\mathsf{Comp}_w^{\Psi}$) $\rightarrow$ ($\mathsf{EEmb}_w^{\Psi}$) $\rightarrow$ ($\mathsf{Emb}_w^{\Psi}$).

A *model complete* theory is a theory which has the property that all embeddings between its models are elementary. So if we choose a model complete base theory then ($\mathsf{EEmb}_w$) and ($\mathsf{Emb}_w$) coincide. To give examples of model complete base theories note first that every theory which allows quantifier elimination (QE) is model complete (cf. [12], Theorem 7.3.1).

*Example 13.* The following theories have QE and are therefore model complete.

(1) Presburger arithmetic with congruence modulo $n$ ($\equiv_n$), $n = 2, 3, ...$ ([6], p.197).
(2) Rational linear arithmetic in the signature $\{+, 0, \leq\}$ ([28]).
(3) Real closed ordered fields ([12], 7.4.4), e.g., the real numbers.
(4) Algebraically closed fields ([5], Ex. 3.5.2; Remark. p. 204; [12], Ch. 7.4, Ex. 2).
(5) Finite fields ([12], Ch. 7.4, Example 2).
(6) The theory of acyclic lists in the signature $\{\mathrm{car}, \mathrm{cdr}, \mathrm{cons}\}$ ([17, 9]).

Not all model complete theories allow QE: the theory of real closed fields (without $<$) is model complete but does not admit quantifier elimination (cf. [5], 3.5.19, and the subsequent remark on p. 204).

**Theorem 14.** *Let $\mathcal{T}_0$ be a $\Pi_0$-theory, $\Pi = (\Sigma_0 \cup \Sigma, \mathsf{Pred})$ and let $\mathcal{K}$ be a set of universally closed, linear and quasi-flat clauses in the signature $\Pi$ and let $\Psi_{\mathcal{K}}$ be a term closure operator with the property that for every flat set of ground terms $T$, $\Psi(T)$ is flat.*

*(1) If the extension $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}$ satisfies ($\mathsf{Emb}_w^{\Psi}$) then it satisfies ($\mathsf{Loc}^{\Psi}$).*
*(2) If the extension $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}$ satisfies ($\mathsf{EEmb}_w^{\Psi}$) then it satisfies ($\mathsf{ELoc}^{\Psi}$).*

*Proof*: (1) Assume that $\mathcal{T}_0 \cup \mathcal{K}$ satisfies condition ($\mathsf{Emb}_w^{\Psi}$) but is not a $\Psi$-local extension of $\mathcal{T}_0$. Then there exists a set $G$ of ground clauses (with additional constants) such that $\mathcal{T}_0 \cup \mathcal{K} \cup G \models \perp$ but $\mathcal{T}_0 \cup \mathcal{K}[\Psi_{\mathcal{K}}(G)] \cup G$ has a weak partial model $\mathcal{P}$ in which all terms in $\Psi_{\mathcal{K}}(G)$ are defined. We assume w.l.o.g. that $G = G_0 \cup G_1$, where $G_0$ contains no function symbols in $\Sigma$ and $G_1$ consists of ground unit clauses of the form $f(c_1, \dots, c_n) \approx c$, where $c_i, c$ are constants in $\Sigma_0 \cup C$ and $f \in \Sigma$.

We construct another structure, $\mathcal{A}$, having the same support as $\mathcal{P}$, which inherits the definitions for all relations in $\mathsf{Pred}$ and all functions in $\Sigma_0 \cup C$ from

$\mathcal{P}$, but on which the domains of definition of the $\Sigma$-functions are restricted as follows: for every $f \in \Sigma$, $f_{\mathcal{A}}(a_1, \dots, a_n)$ is defined if and only if there exist constants $c^1, \dots, c^n$ such that $f(c^1, \dots, c^n)$ is in $\Psi_{\mathcal{K}}(G)$ and $a^i = c^i_{\mathcal{P}}$ for all $i \in \{1, \dots, n\}$. In this case we define $f_{\mathcal{A}}(a_1, \dots, a_n) := f_{\mathcal{P}}(c^1_{\mathcal{P}}, \dots, c^n_{\mathcal{P}})$. The reduct of $\mathcal{A}$ to $(\Sigma_0 \cup C, \mathsf{Pred})$ coincides with that of $\mathcal{P}$. Thus, $\mathcal{A}$ is a model of $\mathcal{T}_0 \cup G_0$. By the way the operations in $\Sigma$ are defined in $\mathcal{A}$ it is clear that $\mathcal{A}$ satisfies $G_1$, so $\mathcal{A}$ satisfies $G$.

To show that $\mathcal{A} \models_w \mathcal{K}$ we use the fact that if $D$ is a clause in $\mathcal{K}$ and $\beta : X \to A$ is an assignment in which $\beta(t)$ is defined for every term $t$ occurring in $D$, then (by the way $\Sigma$-functions are defined in $\mathcal{A}$) we can construct a substitution $\sigma$ with $\sigma(D) \in \mathcal{K}[G]$ and $\beta \circ \sigma = \beta$. As $(\mathcal{P}, \beta) \models_{\mathrm{w}} \sigma(D)$ we can infer $(\mathcal{A}, \beta) \models_{\mathrm{w}} D$.

We now show that $\mathcal{D}(\mathcal{A}) = \{f(a_1, \dots, a_n) \mid f_{\mathcal{A}}(a_1, \dots, a_n) \text{ defined}\}$ is closed under $\Psi_{\mathcal{K}}$. By definition, $f(a_1, \dots, a_n) \in \mathcal{D}(\mathcal{A})$ if and only if there exist $c^1, \dots, c^n$ with $c^i_{\mathcal{A}} = a_i$ for all $i$ and $f(c^1, \dots, c^n) \in \Psi_{\mathcal{K}}(G)$. Thus,

$$\begin{aligned}
\mathcal{D}(\mathcal{A}) &= \{f(a_1, \dots, a_n) \mid f_{\mathcal{A}}(a_1, \dots, a_n) \text{ defined}\} \\
&= \{f(c^1_{\mathcal{A}}, \dots, c^n_{\mathcal{A}}) \mid c^i \text{ constants with } f(c^1, \dots, c^n) \in \Psi_{\mathcal{K}}(G) \text{ and } c^1_{\mathcal{A}} = a_i\} \\
&= \overline{h}(\Psi_{\mathcal{K}}(G)) \text{ where } h(c^i) = a_i \text{ for all } i
\end{aligned}$$

This in turn implies

$$\begin{aligned}
\bar{h}(\Psi_{\mathcal{K}}(\mathcal{D}(\mathcal{A}))) &= \bar{h}(\Psi_{\mathcal{K}}(\bar{h}(\Psi_{\mathcal{K}}(G)))) \\
&= \Psi_{\bar{h}\mathcal{K}}(\bar{h}\bar{h}(\Psi_{\mathcal{K}}(G))) && \text{by property (4) of } \Psi \\
&= \Psi_{\bar{h}\mathcal{K}}(\bar{h}(\Psi_{\mathcal{K}}(G))) \\
&= \bar{h}(\Psi_{\mathcal{K}}(\Psi_{\mathcal{K}}(G))) && \text{by property (4) of } \Psi \\
&\subseteq \bar{h}(\Psi_{\mathcal{K}}(G)) && \text{by property (3) of } \Psi \\
&= \mathcal{D}(\mathcal{A}).
\end{aligned}$$

Now, let $f(\bar{t}) \in \Psi_{\mathcal{K}}(\mathcal{D}(\mathcal{A}))$. It follows that $f(\bar{h}\bar{t}) \in \mathcal{D}(\mathcal{A})$ which implies that $f(\bar{t})$ is defined in $\mathcal{A}$.

As $\mathcal{A} \models_w \mathcal{K}$, $\mathcal{A}$ weakly embeds into a total algebra $\mathcal{B}$ satisfying $\mathcal{T}_0 \cup \mathcal{K}$. But then $\mathcal{B} \models G$, so $\mathcal{B} \models \mathcal{T}_0 \cup \mathcal{K} \cup G$, which is a contradiction.

(2) The proof is similar to the proof for (1) with the following differences. In this case $\mathcal{K}$ consists of augmented clauses, and $G$ consists of augmented ground clauses, i.e. it is of the form $\Phi_0 \cup G_1$, where $\Phi_0$ is a $\Pi_0$-sentence and $G_1$ consists of ground unit clauses of the form $f(c_1, \dots, c_n) \approx c$, where $c_i, c$ are constants in $\Sigma_0 \cup C$ and $f \in \Sigma$. We proceed again by contradiction, and start with a weak partial model $\mathcal{P}$ of $\mathcal{T}_0 \cup \mathcal{K}[\Psi_{\mathcal{K}}(G)] \cup G$ in which all terms in $\Psi_{\mathcal{K}}(G)$ are defined. We construct the structure $\mathcal{A}$ as before. Since the reduct of $\mathcal{A}$ to $(\Sigma_0 \cup C, \mathsf{Pred})$ coincides with that of $\mathcal{P}$, $\mathcal{A}$ is a model of $\mathcal{T}_0 \cup \Phi_0$, so $\mathcal{A}$ satisfies $G$.

We show that $\mathcal{A} \models_w \mathcal{K}$ and that all terms in $\Psi_{\mathcal{K}}(\mathcal{D}(\mathcal{A}))$ are defined as above. By condition $(\mathsf{EEmb}^{\Psi}_w)$, $\mathcal{A}$ weakly embeds into a total algebra $\mathcal{B}$ satisfying $\mathcal{T}_0 \cup \mathcal{K}$, such that $\mathcal{A}|_{\Pi_0}$ elementarily embeds into $\mathcal{B}|_{\Pi_0}$. Then $\mathcal{B} \models G_1$ and since

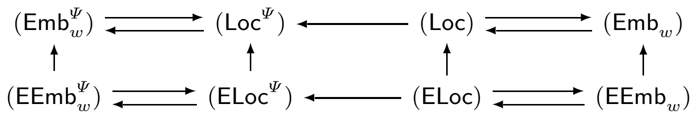



**Fig. 1.** Relations between locality and embeddability

elementary embeddings preserve the truth of arbitrary formulae, $\Phi_0$ is true also in $\mathcal{B}$. Thus, $\mathcal{B} \models \mathcal{T}_0 \cup \mathcal{K} \cup G$, which is a contradiction. □

The results above give us the following relations (see Figure 1) between these properties under the conditions on $\Psi$ and $\mathcal{K}$ in the statements of Theorem 14 and 12. Note that because of Definition 11.(1), $(\mathsf{Loc})$ implies $(\mathsf{Loc}^{\Psi})$.

The results naturally adapt to yield links between $(\mathsf{Loc}^{\Psi}_f)$ and $(\mathsf{Emb}^{\Psi}_{w,f})$, and $(\mathsf{ELoc}^{\Psi}_f)$ and $(\mathsf{EEmb}^{\Psi}_{w,f})$ respectively.

**Comments.** We can generalize these results even further and refer to versions of locality resp. embeddability parameterized by a fragment $\mathcal{F}$ of the theory $\mathcal{T}_0$ (containing the ground clause fragment): In condition $(\mathsf{EEmb}^{\Psi}_w(\mathcal{F}))$ we require that every $\mathcal{A} \in \mathsf{PMod}^{\Psi}_w(\Sigma, \mathcal{T})$ weakly embeds into a total model of $\mathcal{T}$ such that the restriction of the embedding on $\Pi_0$ preserves the truth of formulae in $\mathcal{F}$, possibly with parameters in $\mathcal{A}$, and by allowing in $(\mathsf{ELoc}^{\Psi}_{\mathcal{F}})$ that all clauses and the goal are $\mathcal{F}$-augmented clauses (cf. Section 2). We do not present these extensions in detail here due to space constraints.

*Remark 15.* Theorem 14 allows us to further extend the $\Psi$-locality results established in [15] for the fragment of the theory of arrays introduced in [3] to a fragment of the theory of arrays consisting of conjunctions of the following types of formulae:

- arbitrary ground sentences
- arbitrary sentences in the signature of the theory of elements
- array property formulae of the form:

$$\forall \overline{i}(\phi_i(\overline{i}) \rightarrow \phi_V(\overline{i}))$$

  where $\phi_i(\overline{i})$ is an index guard as defined in [3], and $\phi_V(\overline{i})$ is a formula of the form $\psi(a_1(\overline{i}_1), \ldots, a_k(\overline{i}_k))$, where $\overline{i}_1, \ldots, \overline{i}_k$ contain subsets of the set of variable $\overline{i}$ and $\psi(z_1, \ldots, z_k)$ is an arbitrary formula in the signature of the theory of elements whose only free variables are $z_1, \ldots, z_k$. As in [3], we require that every index variable in $\overline{i}$ occurring in $\phi_V(\overline{i})$ occurs below some array read and that nested array reads are not allowed.

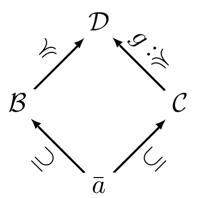
Then the $\Psi$-locality proof in [3] which uses the fact that any partial model of the set of formulae can be completed to a total model without changing its support can be naturally extended to this more general class of formulae and can be used to prove $\Psi$-locality also in this case.

## 5 Locality transfer results

In this section we present some locality transfer results.

We use the following theorem which can be regarded as a generalization of Robinson's joint consistency theorem.

**Theorem 16 ([12], 5.5.1).** *Let $\Pi_1, \Pi_2$ be signatures, $\Pi = \Pi_1 \cap \Pi_2$, $\mathcal{B}$ a $\Pi_1$-structure, $\mathcal{C}$ a $\Pi_2$-structure and $\bar{a}$ a sequence of elements in both $\mathcal{B}$ and $\mathcal{C}$ such that $(\mathcal{B}|_\Pi,\ \bar{a}) \equiv (\mathcal{C}|_\Pi,\ \bar{a})$. Then there is a $(\Pi_1 \cup \Pi_2)$-structure $\mathcal{D}$ such that $\mathcal{B} \preccurlyeq \mathcal{D}|_{\Pi_1}$ and an elementary embedding $g : \mathcal{C} \to \mathcal{D}|_{\Pi_2}$ with $g(a_i) = a_i$ for every $a_i$ in $\overline{a}$.*



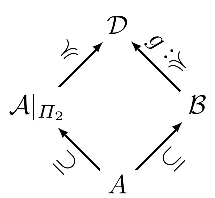

*if* $(\mathcal{B}|_\Pi,\ \bar{a}) \equiv (\mathcal{C}|_\Pi,\ \bar{a})$

An application of Theorem 16 is the transfer of elementary embeddings.

**Theorem 17 ((EEmb) Transfer).** *Let $\Pi_0 = (\Sigma_0, \mathsf{Pred})$ be a signature, $\mathcal{T}_0$ a theory in $\Pi_0$, $\Sigma_1$ and $\Sigma_2$ two disjoint sets of new function symbols, $\Pi_i := (\Sigma_0 \cup \Sigma_i, \mathsf{Pred})$, $i = 1, 2$. Assume that $\mathcal{T}_2$ is a $\Pi_2$-theory with $\mathcal{T}_0 \subseteq \mathcal{T}_2$, and $\mathcal{K}$ is a set of universally closed $\Pi_1$-clauses. If the extension $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}$ enjoys $(\mathsf{EEmb}_w)$ then so does the extension $\mathcal{T}_2 \subseteq \mathcal{T}_2 \cup \mathcal{K}$. In particular, if $\mathcal{K}$ is (quasi)-flat and linear then extension $\mathcal{T}_2 \subseteq \mathcal{T}_2 \cup \mathcal{K}$ satisfies condition $(\mathsf{ELoc})$.*

*If all variables in clauses in $\mathcal{K}$ occur below $\Sigma_1$-functions, and ground satisfiability is decidable in $\mathcal{T}_2$, then ground satisfiability is decidable in $\mathcal{T}_2 \cup \mathcal{K}$.*

*Proof*: Let $\mathcal{A} \in \mathsf{PMod}_w(\Sigma_1, \mathcal{T}_2 \cup \mathcal{K})$. We need to show that $\mathcal{A}$ embeds into a total model $\mathcal{D}$ of $\mathcal{T}_2 \cup \mathcal{K}$ such that $\mathcal{A}|_{\Pi_2}$ is elementarily embedded into $\mathcal{D}|_{\Pi_2}$. By assumption, $\mathcal{A}$ is a total model of $\mathcal{T}_2$ and therefore of $\mathcal{T}_0$. It follows that there is a (total) model $\mathcal{B}$ of $\mathcal{T}_0 \cup \mathcal{K}$ such that $\mathcal{A}$ is a weak substructure of $\mathcal{B}$ and $\mathcal{A}|_{\Pi_0} \preccurlyeq \mathcal{B}|_{\Pi_0}$. Let $\bar{a}$ list all the elements of $\mathcal{A}$, then it holds that $(\mathcal{A}|_{\Pi_0}, \bar{a}) \equiv (\mathcal{B}|_{\Pi_0}, \bar{a})$. We use Theorem 16 to get a $(\Pi_1 \cup \Pi_2)$-structure $\mathcal{D}$ such that $\mathcal{A}|_{\Pi_2} \preccurlyeq \mathcal{D}|_{\Pi_2}$ (in particular $\mathcal{D} \models \mathcal{T}_2$) and an elementary embedding $g : \mathcal{B} \to \mathcal{D}|_{\Pi_1}$ with $g(a) = a$ for all elements $a \in A$.




with $(\mathcal{A}|_{\Pi_0},\ A) \equiv (\mathcal{B}|_{\Pi_0},\ A)$

The only thing left to show is that $\mathcal{A}$ is a weak substructure of $\mathcal{D}$. Let $f \in \Sigma_1$ be an extension function such that $f_\mathcal{A}(\bar{a})$ is defined and equal to $b \in A$, say. It follows that $f_{\mathcal{A}}(\bar{a}) = f_{\mathcal{B}}(\bar{a}) = b$ because $\mathcal{A}|_{\Pi_1}$ is a weak substructure of $\mathcal{B}$. Because the diagram commutes and $g(a) = a$, for all $a \in A$, we have $f_{\mathcal{D}}(\bar{a}) = f_{\mathcal{D}}(g\bar{a}) = g(f_{\mathcal{B}}(\bar{a})) = g(b) = b = f_{\mathcal{A}}(\bar{a})$. □

The result extends in a natural way to the case ($\mathsf{EEmb}_{w,f}$), i.e. the embeddability of models where the domains of the extension functions are all finite. Theorem 17 is a very useful result, which allows us to identify a large number of local extensions. We illustrate its applicability on one example.

*Example 18.* Let $\mathsf{Lat}$ be the theory of lattices and $\mathcal{T}_1 = \mathsf{Lat} \cup \mathsf{Mon}_f$, where $\mathsf{Mon}_f = \{\forall x, y\ (x \leq y \rightarrow f(x) \leq f(y))\}$ is the monotonicity of a new function symbol $f$. Using techniques similar to the ones used in [25] we can prove that the extension $\mathsf{Lat} \subseteq \mathsf{Lat} \cup \mathsf{Mon}_f$ satisfies condition ($\mathsf{Comp}_{w,f}$) hence also ($\mathsf{EEmb}_{w,f}$). Let $\mathcal{T}$ be any extension of the theory of lattices with signature not containing $f$ (this can be the theory of distributive lattices, Heyting algebras, Boolean algebras, any theory with a total order – e.g. the (ordered) theory of integers or of reals, etc.). By Theorem 17, $\mathcal{T} \subseteq \mathcal{T} \cup \mathsf{Mon}_f$ satisfies condition ($\mathsf{EEmb}_{w,f}$), hence the extended locality condition ($\mathsf{ELoc}_f$).

For model complete base theories Theorem 17 specializes as follows.

**Corollary 19.** *Let $\Pi_0$ be a signature, $\mathcal{T}_0$ a model complete theory in $\Pi_0$, and $\Sigma_1$ and $\Sigma_2$ two disjoint sets of new function symbols. Let $\Pi_i = (\Sigma_0 \cup \Sigma_i, \mathsf{Pred})$, $i = 1, 2$. Let $\mathcal{K}$ be a set of flat and linear $\Pi_1$-clauses and $\mathcal{T}_2$ be an arbitrary $\Pi_2$-theory with $\mathcal{T}_0 \subseteq \mathcal{T}_2$. If the extension $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}$ is local then the extension $\mathcal{T}_2 \subseteq \mathcal{T}_2 \cup \mathcal{K}$ is local as well.*

### 5.1 Locality and model completeness

A model complete theory can sometimes be regarded as the completion of another theory with the same universal fragment. Recall that the *diagram* of a first-order structure $\mathcal{A}$ is the set of all ground literals true in the extension $(\mathcal{A}, A)$ of $\mathcal{A}$ where we have a constant for each element of $A$.

**Definition 20.** *A theory $\mathcal{T}^*$ is called a* model completion *of $\mathcal{T}$ if (i) $\mathcal{T}$ and $\mathcal{T}^*$ are cotheories (i.e. every model of $\mathcal{T}$ can be extended to a model of $\mathcal{T}^*$ and vice versa), (ii) $\mathcal{T}^*$ is model complete and (iii) for every model $\mathcal{A}$ of $\mathcal{T}$, $\mathcal{T}^* \cup \Delta_A$ is complete where $\Delta_A$ is the diagram of $\mathcal{A}$.*

*Example 21.* Below we present some examples of model completions:

(1) The theory of algebraically closed fields is the model completion of the theory of fields. This was the motivating example for developing the theory of model completions ([5], Examples 3.5.2, 3.5.12; Remark 3.5.6 following; [12], 7.3).
(2) The theory of dense total orders without endpoints is the model completion of the theory of total orders ([9]).
(3) The theory of atomless Boolean algebras is the model completion of Boolean algebras ([5], Example 3.5.12, cf. also p. 196).

(4) Universal Horn theories in finite signatures have a model completion if they are locally finite and have the amalgamation property (e.g., graphs, posets) ([29]).

**Theorem 22.** *Let $\mathcal{T}_0$ be a theory. Assume that $\mathcal{T}_0$ has a model completion $\mathcal{T}_0^*$ such that $\mathcal{T}_0 \subseteq \mathcal{T}_0^*$. Let $\mathcal{T} = \mathcal{T}_0 \cup \mathcal{K}$ be an extension of $\mathcal{T}_0$ with new function symbols $\Sigma$ whose properties are axiomatized by a set of flat and linear clauses $\mathcal{K}$ (all of which contain symbols in $\Sigma$).*

*(1) Assume that:*
*(i) Every model of $\mathcal{T}_0 \cup \mathcal{K}$ embeds[3] into a model of $\mathcal{T}_0^* \cup \mathcal{K}$.*
*(ii) $\mathcal{T}_0 \cup \mathcal{K}$ is a local extension of $\mathcal{T}_0$.*
*Then $\mathcal{T}_0^* \subseteq \mathcal{T}_0^* \cup \mathcal{K}$ satisfies condition* $(\mathsf{EEmb}_w)$*, hence if $\mathcal{K}$ is a set of quasi-flat and linear augmented clauses also condition* $(\mathsf{ELoc})$ *as extension of $\mathcal{T}_0^*$.*
*(2) If all variables in $\mathcal{K}$ occur below an extension function and $\mathcal{T}_0^* \cup \mathcal{K}$ is a local extension of $\mathcal{T}_0^*$ then $\mathcal{T}_0 \cup \mathcal{K}$ is a local extension of $\mathcal{T}_0$.*

*Proof*: (1) Let $\mathcal{A} \in \mathsf{PMod}_w(\Sigma, \mathcal{T}_0^* \cup \mathcal{K})$. Since we assumed that $\mathcal{T}_0 \subseteq \mathcal{T}_0^*$, $\mathcal{A} \in \mathsf{PMod}_w(\Sigma, \mathcal{T}_0 \cup \mathcal{K})$. By locality, $\mathcal{A}$ weakly embeds into a total model $\mathcal{A}_1$ of $\mathcal{T}_0 \cup \mathcal{K}$. By compatibility, $\mathcal{A}_1$ embeds into a model of $\mathcal{T}_0^* \cup \mathcal{K}$. Since the weak embedding between $\mathcal{A}$ and $\mathcal{A}_1$ restricts to an embedding $i : \mathcal{A}|_{\Pi_0} \to \mathcal{A}_1|_{\Pi_0}$, and $\mathcal{T}_0^*$ is model complete, $i$ is an elementary embedding. Thus, condition $(\mathsf{EEmb}_w)$ holds.

(2) We show $(\mathsf{Emb}_w)$. Let $\mathcal{A} \in \mathsf{PMod}_w(\Sigma, \mathcal{T}_0 \cup \mathcal{K})$. Then $\mathcal{A}|_{\Pi_0}$ is a model of $\mathcal{T}_0$, and it therefore embeds (with an embedding $i_1$) into a model $\mathcal{B}$ of $\mathcal{T}_0^*$. By the same argument as in Lemma 27 (Sect. 6.2), we can transfer the partial functions from $\mathcal{A}$ to $\mathcal{B}$ using $i_1$ such that $\mathcal{B}$ weakly satisfies the clauses in $\mathcal{K}$ and $i_1$ is a weak embedding. Then $\mathcal{B} \in \mathsf{PMod}_w(\Sigma, \mathcal{T}_0^* \cup \mathcal{K})$. By the locality of the extension $\mathcal{T}_0^* \subseteq \mathcal{T}_0^* \cup \mathcal{K}$, $\mathcal{B}$ weakly embeds (by a weak embedding $i_2$) into a total model $\mathcal{C}$ of $\mathcal{T}_0^* \cup \mathcal{K}$, hence of $\mathcal{T}_0 \cup \mathcal{K}$ (we assumed that $\mathcal{T}_0 \subseteq \mathcal{T}_0^*$). Thus, the composition of $i_1$ and $i_2$ is a weak embedding of $\mathcal{A}$ into a total model $\mathcal{C}$ of $\mathcal{T}_0 \cup \mathcal{K}$. □

The results extend in a natural way to $\Psi$-locality and to finite versions of embeddability and locality.

*Example 23.* We show that the extension of the theory $\mathsf{TOrd}$ of total orderings with a strictly monotone function, i.e. a function $f$ satisfying the axiom:

$$\mathsf{SMon}(f) \quad \forall x, y (x < y \to f(x) < f(y))$$

satisfies condition $(\mathsf{Loc}_f)$. To show this, note that the model completion $\mathsf{TOrd}^*$ of $\mathsf{TOrd}$ is the theory of dense total orderings without endpoints. We show that the extension $\mathsf{TOrd}^* \subseteq \mathsf{TOrd}^* \cup \mathsf{SMon}(f)$ satisfies condition $(\mathsf{ELoc}_f)$. Indeed, let $\mathcal{A} = (A, \leq, f)$ be a partial model of $\mathsf{TOrd}^* \cup \mathsf{SMon}(f)$ where the domain of definition of $f$ is finite, say $\{a_1, \dots, a_n\} \subseteq A$ where $a_1 < a_2 < \cdots < a_n$. W.l.o.g. we can assume that $A$ is countable. Let $b_i = f(a_i) \in A$, $1 \leq i \leq n$. Then $b_1 < b_2 < \cdots < b_n$. Let $A_0 = \{x \in A \mid x < a_1\}$, $A_i = \{x \in A \mid a_i < x < a_{i+1}\}$,

[3] If $\mathcal{T}_0$ is universal, this is the notion of compatibility defined in [9].

for $1 \leq i \leq n-1$, and $A_n = \{x \in A \mid a_n < x\}$, and $B_0 = \{x \in A \mid x < b_1\}, B_i = \{x \in A \mid b_i < x < b_{i+1}\}$, for $1 \leq i \leq n-1$, and $B_n = \{x \in A \mid b_n < x\}$. All these sets are countable models of $\mathsf{TOrd}^*$ hence isomorphic (since the theory of dense total orderings without endpoints is $\omega$-categorical). These isomorphisms can be used to extend the partial map to a strictly monotone map from $A$ to $A$.

Using Theorem 22(2) we can conclude that the extension $\mathsf{TOrd} \subseteq \mathsf{TOrd} \cup \mathsf{SMon}(f)$ satisfies condition $(\mathsf{ELoc}_f)$.

*Example 24.* Similarly, we can prove that the two-sorted extension of the pure theory of equality (with sorts $a$ and $b$) with a function $f$ (with arity $a \rightarrow b$) satisfying $\mathsf{Inj}(f)\ \forall x, y\, (x \neq y \rightarrow f(x) \neq f(y))$ is local.

To show this, note that the model completion $\mathsf{Eq}^*$ of the pure theory of equality $\mathsf{Eq}$ is the theory of infinite sets. Let $\mathsf{Eq}^* \dot\sqcup \mathsf{Eq}^*$ be the disjoint union of two copies of the theory of equality, one of sort $a$ and one of sort $b$. We show that the extension $\mathsf{Eq}^* \dot\sqcup \mathsf{Eq}^* \subseteq (\mathsf{Eq}^* \dot\sqcup \mathsf{Eq}^*) \cup \mathsf{Inj}(f)$ satisfies condition $(\mathsf{ELoc}_f)$. Indeed, let $\mathcal{A} = (A_a, A_b, f)$ be a partial model of $(\mathsf{Eq}^* \dot\sqcup \mathsf{Eq}^*) \cup \mathsf{Inj}(f)$ where the domain of definition of $f$ is finite, say $\{a_1, \dots, a_n\} \subseteq A_a$. W.l.o.g. we can assume that $A_a$, $A_b$ are countable. Let $b_i = f(a_i) \in A_b$, $1 \leq i \leq n$. Since $\{a_i\}_{i=1,m}$ are all distinct, $\{b_i\}_{i=1,m}$ are all distinct. The sets $A_a \backslash \{a_1, \dots, a_n\}$ and $A_b \backslash \{b_1, \dots, b_n\}$ have the same cardinality, so there exists an injective map between these two sets which can be used to extend $f$.

Using Theorem 22(2) we conclude that the extension $(\mathsf{Eq} \dot\sqcup \mathsf{Eq}) \cup \mathsf{Inj}(f) \subseteq (\mathsf{Eq} \dot\sqcup \mathsf{Eq}) \cup \mathsf{Inj}(f)$ satisfies condition $(\mathsf{ELoc}_f)$.

## 6 Combinations of local theories

We now identify situations in which the union of two local extensions of a common base theory is again a local extension of the base theory. This was first studied in [23] and [24]. Here, we extend some of the results in [23] and [24] by using instead of the completability of partial models the condition $(\mathsf{EEmb}_w)$, and also embeddability conditions parameterized by term closure operators.

### 6.1 Case 1: Both theories satisfy $(\mathsf{EEmb}_w)$

We first show that extended locality is preserved when combining theories.

**Lemma 25.** *Let $\Pi_0$ be a signature, $\mathcal{T}_0$ a $\Pi_0$-theory, $\Sigma_1$ and $\Sigma_2$ two disjoint sets of fresh function symbols and $\mathcal{K}_i$ a set of universally closed $\Pi_i$-clauses (where $\Pi_i = (\Sigma_0 \cup \Sigma_i, \mathsf{Pred})$), for $i = 1, 2$. If both extensions $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}_i$, $i = 1, 2$, satisfy $(\mathsf{EEmb}_w)$ then so does the extension $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}_1 \cup \mathcal{K}_2$. If $\mathcal{K}_1 \cup \mathcal{K}_2$ is (quasi)-flat and linear then the extension $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}_1 \cup \mathcal{K}_2$ is local.*

*Proof*: Let $\mathcal{A} \in \mathsf{PMod}_w(\Sigma_1 \cup \Sigma_2, \mathcal{T}_0 \cup \mathcal{K}_1 \cup \mathcal{K}_2)$. By assumption there are (total) models $\mathcal{B}, \mathcal{C}$ of $\mathcal{T}_0 \cup \mathcal{K}_1$ and $\mathcal{T}_0 \cup \mathcal{K}_2$ respectively into which $\mathcal{A}$ weakly embeds. We may assume w.l.o.g. that $\mathcal{A}$ is a weak substructure of both $\mathcal{B}$ and $\mathcal{C}$. By assumption we have $(\mathcal{A}|_{\Pi_0}, \bar{a}) \equiv (\mathcal{B}|_{\Pi_0}, \bar{a}) \equiv (\mathcal{C}|_{\Pi_0}, \bar{a})$, where $\bar{a}$ lists all elements of $\mathcal{A}$.

We use Theorem 16 to obtain a $(\Pi_1 \cup \Pi_2)$-structure $\mathcal{D}$ such that $\mathcal{B} \preccurlyeq \mathcal{D}|_{\Pi_1}$ and an elementary embedding $\psi : \mathcal{C} \to \mathcal{D}|_{\Pi_2}$ with $\psi(a) = a$ for all elements $a \in A$. Now $\mathcal{A}$ is a weak substructure of $\mathcal{D}$: Let $f$ be an extension function such that $f_{\mathcal{A}}(\bar{a})$ is defined. If $f$ is in $\Sigma_1$ we have $f_{\mathcal{A}}(\bar{a}) = f_{\mathcal{B}}(\bar{a}) = f_{\mathcal{D}}(\bar{a})$. If $f$ is an $\Sigma_2$-function it follows that $f_{\mathcal{A}}(\bar{a}) = \psi(f_{\mathcal{A}}(\bar{a})) = \psi(f_{\mathcal{C}}(\bar{a})) = f_{\mathcal{D}}(\psi\bar{a}) = f_{\mathcal{D}}(\bar{a})$. Obviously, $\mathcal{D}$ is a model of $\mathcal{T}_1 \cup \mathcal{T}_2$ and $(\mathcal{A}|_{\Pi_0}, \bar{a}) \equiv (\mathcal{D}|_{\Pi_0}, \bar{a})$. So $\mathcal{D}$ is as desired. □

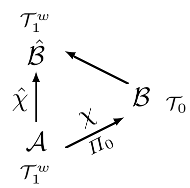
If $\mathcal{T}_0$ is a model complete base theory then $(\mathsf{EEmb}_w)$ and $(\mathsf{Emb}_w)$ coincide.

**Corollary 26.** *Let $\mathcal{T}_0$ be a model complete $\Pi_0$-theory, $\Sigma_1$ and $\Sigma_2$ two disjoint sets of fresh function symbols and $\mathcal{K}_i$ a set of universally closed $\Pi_i$-clauses for $i = 1, 2$. If both extensions $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}_i$, $i = 1, 2$, satisfy $(\mathsf{Emb}_w)$ then so does the extension $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}_1 \cup \mathcal{K}_2$. In particular, if $\mathcal{K}_1 \cup \mathcal{K}_2$ is (quasi)-flat and linear then the extension $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}_1 \cup \mathcal{K}_2$ is local.*

### 6.2 Case 2: One theory satisfies $(\mathsf{EEmb}_w)$

We consider combinations of theory extensions among which one satisfies condition $(\mathsf{EEmb}_w)$. We extend Theorem 19 of [24] (cf. also [23]) to handle this situation. We first start by presenting a structure transfer lemma proved in [23].

**Lemma 27 ([23]).** *Let $\mathcal{T}_0$ be a theory in $\Pi_0$, $\Sigma_1$ a set of fresh function symbols and $\mathcal{K}$ a flat set of clauses in $\Pi_0 \cup \Sigma_1$. Let $\mathcal{T}_1 := \mathcal{T}_0 \cup \mathcal{K}$ and assume that for each clause $C$ of $\mathcal{K}$ it holds that each variable appears below a $\Sigma_1$-function. Let $\mathcal{A} \in \mathsf{PMod}_w(\Sigma_1, \mathcal{T}_1)$ and let $\mathcal{B}$ be a (total) model of $\mathcal{T}_0$ such that $\chi : \mathcal{A}|_{\Pi_0} \to \mathcal{B}$ is a $\Pi_0$-embedding. Then $\chi$ and $\mathcal{B}$ can be extended such that $\hat{\chi} : \mathcal{A} \to \hat{\mathcal{B}}$ is a weak $\Pi_1$-embedding and $\hat{\mathcal{B}} \in \mathsf{PMod}_w(\Sigma_1, \mathcal{T}_1)$.*
*Schematically:*




*where $\mathcal{T}_1^w$ indicates that we have a weak partial model of $\mathcal{T}_1$.*

*Proof*: We include here the proof given in [23].
For every $b_1, \dots, b_n \in B$ and every $f \in \Sigma_1$ define

$$f_{\mathcal{B}}(b_1, \dots, b_n) := \begin{cases} b & \text{if } \exists a_1, \dots, a_n \in A \text{ such that all } b_i = \chi(a_i), \\ & f_A(a_1, \dots, a_n) \text{ is defined in } \mathcal{A}, \\ & \text{and } b = \chi(f_A(a_1, \dots, a_n)) \\ \text{undefined} & \text{otherwise.} \end{cases}$$

As $\chi$ is injective, $f_{\mathcal{B}}$ is well-defined. By hypothesis, $\chi$ is a $\Pi_0$-embedding. With the definition of operations in $\Sigma_1$ given above, $\chi$ is also a weak $\Sigma_1$-homomorphism. Let $a_1, \dots, a_n \in A$ and $f \in \Sigma_1$ be such that $f_{\mathcal{A}}(a_1, \dots, a_n)$ is defined. Then, by the definition of $f_{\mathcal{B}}$, $f_{\mathcal{B}}(\chi(a_1), \dots, \chi(a_n))$ is defined and equal to $\chi(f_{\mathcal{A}}(a_1, \dots, a_n))$.

We now prove that with the operations defined as shown before $\mathcal{B}$ weakly satisfies $\mathcal{K}$. Let $C \in \mathcal{K}$ and let $\beta : X \to B$ be an assignment of elements in $B$ to the variables in $C$. Assume that for every term $t$ occurring in $C$, $\beta(t)$ is defined in $\mathcal{B}$ (otherwise, due to the definition of weak satisfiability, $(\mathcal{B}, \beta) \models_w C$ trivially). In order to show that $(\mathcal{B}, \beta) \models_w C$, we construct an assignment $\alpha$ of elements in $A$ to the variables in $C$, and use the fact that $(\mathcal{A}, \alpha) \models_w C$.

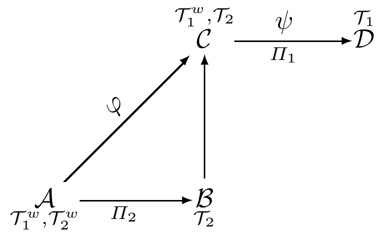
Let $t = f(t_1, \dots, t_k)$ be an arbitrary term occurring in $C$, with $f \in \Sigma_1$. As $\beta(t)$ is defined, $f_{\mathcal{B}}(\beta(t_1), \dots, \beta(t_k))$ is defined in $\mathcal{B}$, hence there exist $a_1, \dots, a_k \in A$ such that $\chi(a_i) = \beta(t_i)$, $f_A(a_1, \dots, a_k)$ is defined, and $f_{\mathcal{B}}(\beta(t_1), \dots, \beta(t_k)) = \chi(f_A(a_1, \dots, a_n))$. As all clauses in $\mathcal{K}$ are $\Sigma_1$-flat, all terms $t_i$ are variables. In this way we can associate with every variable $x$ occurring as argument in a term $f(t_1, \dots, t_n)$ of $C$ with $f \in \Sigma_1$ an element $a_x \in A$ such that $\chi(a_x) = \beta(x)$. Assume that for some such (variable) subterm $x$, two elements of $A$, say $a_x$ and $a'_x$, can be associated in this way. Then $\chi(a_x) = \beta(x) = \chi(a'_x)$, and the injectivity of $\chi$ guarantees that $a_x = a'_x$. This shows that an assignment $\alpha : X \to A$ can be defined, such that for all variables in $C$ occurring below a function symbol in $\Sigma$ (hence for all variables in $C$) $\alpha(x) := a_x$. It is easy to see that for every term $t$ occurring in $C$, $\chi(\alpha(t)) = \beta(t)$. As $(\mathcal{A}, \alpha) \models C$ and $\chi$ is a weak $\Pi$-embedding it follows that $(\mathcal{B}, \beta) \models C$. □

**Theorem 28.** *Let $\mathcal{T}_0$ be a theory in the signature $\Pi_0$, $\Sigma_1$ and $\Sigma_2$ two disjoint sets of new function symbols, and $\Pi_i = (\Sigma_0 \cup \Sigma_i, \mathsf{Pred})$. Let $\mathcal{K}_i$ be a set of $\Pi_i$-clauses for $i = 1, 2$, and $\mathcal{T}_i := \mathcal{T}_0 \cup \mathcal{K}_i$, $i = 1, 2$. Assume that:*

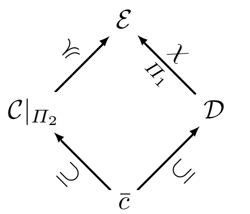
*(1) $\mathcal{T}_0 \subseteq \mathcal{T}_1$ satisfies $(\mathsf{EEmb}_w)$,*
*(2) $\mathcal{T}_0 \subseteq \mathcal{T}_2$ satisfies $(\mathsf{Emb}_w)$ and*
*(3) $\mathcal{K}_1$ is $\Sigma_1$-flat in which all variables are shielded, i.e., all variables occur below some $\Sigma_1$-function.*

*Then the extension $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}_1 \cup \mathcal{K}_2$ satisfies $(\mathsf{Emb}_w)$. If $\mathcal{K}_1 \cup \mathcal{K}_2$ is (quasi)-flat and linear then the extension $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}_1 \cup \mathcal{K}_2$ is local.*

*Proof*: Let $\mathcal{A}$ be a weak partial model of $\mathcal{T}_0 \cup \mathcal{K}_1 \cup \mathcal{K}_2$. Then $\mathcal{A}|_{\Pi_2}$ is a partial model of $\mathcal{T}_2$. By assumption $\mathcal{A}|_{\Pi_2}$ weakly embeds into a total model $\mathcal{B}$ of $\mathcal{T}_2$. By Lemma 27, we can extend $\mathcal{B}|_{\Pi_0}$ to a weak partial model $\mathcal{C}^-$ of $\mathcal{T}_1$. Reattaching $\mathcal{B}$'s $\Sigma_2$-functions thus gives us a model $\mathcal{C}$ of $\mathcal{T}_2$ which is also a partial model of $\mathcal{T}_1$ and some weak embedding $\varphi : \mathcal{A} \to \mathcal{C}$. Now we use $(\mathsf{EEmb}_w)$ to obtain an $\Pi_0$-elementary extension $\mathcal{D}$ of $\mathcal{C}$ which is a total model of $\mathcal{T}_1$. Pictorially:




where $(\mathcal{C}|_{\Pi_0}, \bar{c})_{\bar{c}\in C} \equiv (\mathcal{D}|_{\Pi_0}, \psi\bar{c})_{\bar{c}\in C}$. We may further assume w.l.o.g. that $\psi$ is the identity. Now we use Theorem 16 on $\mathcal{C}|_{\Pi_2}$ and $\mathcal{D}$ (choose some listing $\bar{c}$ of $\mathcal{C}$'s elements) to get a $(\Pi_1 \cup \Pi_2)$-structure $\mathcal{E}$ such that $\mathcal{C}|_{\Pi_2} \preccurlyeq \mathcal{E}|_{\Pi_2}$ and some elementary embedding $\chi : \mathcal{D} \to \mathcal{E}|_{\Pi_1}$.




because $(\mathcal{C}|_{\Pi_0},\ \bar{c}) \equiv (\mathcal{D}|_{\Pi_0},\ \bar{c})$

In particular, $\mathcal{C}|_{\Pi_2} \equiv \mathcal{E}|_{\Pi_2}$ and $\mathcal{E}|_{\Pi_1} \equiv \mathcal{D}$. Hence, $\mathcal{E}$ is a model of $\mathcal{T}_1 \cup \mathcal{T}_2$ and $\varphi$ can be extended to a weak embedding from $\mathcal{A}$ into $\mathcal{E}$: Suppose $f_{\mathcal{A}}(\bar{a})$ is defined. We need to show that $\varphi(f_{\mathcal{A}}(\bar{a})) = f_{\mathcal{E}}(\varphi\bar{a})$. If $f$ is a $\Sigma_2$-function we have

$$\varphi(f_{\mathcal{A}}(\bar{a})) = f_{\mathcal{C}}(\varphi\bar{a}) = f_{\mathcal{E}}(\varphi\bar{a}).$$

If $f$ is a $\Sigma_1$-function we have

$$\varphi(f_{\mathcal{A}}(\bar{a})) = f_{\mathcal{C}}(\varphi\bar{a}) = f_{\mathcal{D}}(\varphi\bar{a})$$

because $\mathcal{C}$ is a weak substructure of $\mathcal{D}$ as a $\Pi_1$-structure. On the other hand, $\chi$ is the identity on $C$, thus,

$$f_{\mathcal{C}}(\varphi\bar{a}) = \chi(f_{\mathcal{C}}(\varphi\bar{a})) = \chi(f_{\mathcal{D}}(\varphi\bar{a})) = f_{\mathcal{E}}(\chi\varphi\bar{a}) = f_{\mathcal{E}}(\varphi\bar{a}). \qquad \square$$

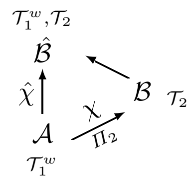
### 6.3 Combinations of $\Psi$-local theory extensions

We now study combinations of $\Psi_i$-local extensions (with different $\Psi_i$'s) over a common base theory. For a partial algebra $\mathcal{A}$ and a term closure operator $\Psi$, let us write $\Psi_{\mathcal{K}}(\mathcal{A})$ for the set $\Psi_{\mathcal{K}}(\mathcal{D}(\mathcal{A}))$ in this section. The following lemma lifts the argument in [24] (cf. also [23]) to $\Psi$-locality.

**Lemma 29.** *Let $\mathcal{T}_0$ be a $\Pi_0$-theory, $\Sigma_1$ and $\Sigma_2$ two disjoint sets of new function symbols, $\Pi_i = (\Sigma_0 \cup \Sigma_i, \mathsf{Pred})$, and $\mathcal{K}_i$ a set of universally closed $\Pi_i$-clauses, for $i = 1, 2$. Let $\mathcal{T}_i := \mathcal{T}_0 \cup \mathcal{K}_i$, $i = 1, 2$. Let $\Psi_{\mathcal{K}_1}$ be a closure operator w.r.t. $(\Pi_1^C)$-terms, $\mathcal{A}$ a $(\Pi_1 \cup \Pi_2)$-structure such that $\mathcal{A}|_{\Pi_1} \in \mathsf{PMod}_w^{\Psi}(\Sigma_1, \mathcal{T}_1)$ and $\mathcal{B}$ a total model of $\mathcal{T}_2$ such that $\chi : \mathcal{A}|_{\Pi_2} \to \mathcal{B}$ is a weak $\Pi_2$-embedding. Assume that:*

*(1)* $\mathcal{K}_1$ *is* $\Sigma_1$*-flat,*
*(2) all variables of* $\mathcal{K}_1$ *appear below an extension function,*
*(3) all terms in* $\Psi_{\mathcal{K}_1}(D(\mathcal{A}|_{\Pi_1}))$ *are defined in* $(\mathcal{A}|_{\Pi_1}, A)$*.*

*Then* $\chi$ *and* $\mathcal{B}$ *can be expanded such that* $\hat{\chi}{:}\mathcal{A}{\to}\hat{\mathcal{B}}$ *is a weak* $(\Pi_1{\cup}\Pi_2)$*-embedding,* $\hat{\mathcal{B}}|_{\Pi_1}{\in}\mathsf{PMod}_w(\Sigma_1, \mathcal{T}_1)$*,* $\hat{\mathcal{B}}|_{\Pi_2} = \mathcal{B}$ *and all terms of* $\Psi_{\mathcal{K}_1}(D(\mathcal{B}|_{\Pi_1}))$ *are defined in* $(\mathcal{B}|_{\Pi_1}, B)$*.*
*Schematically:*




*where* $\mathcal{T}_1^w$ *indicates that we have a weak partial model of* $\mathcal{T}_1$*.*

*Proof.* We need to add the $\Sigma_1$-functions to $\mathcal{B}$. For $f \in \Sigma_1$ set

$$f_{\hat{\mathcal{B}}}(b_1, ..., b_n) := \begin{cases} \chi(f_{\mathcal{A}}(a_1, ..., a_n)) & \text{if } \exists a_i \in A \text{ s. t. } \chi(a_i) = b_i \\ & \text{and } f_{\mathcal{A}}(a_1, ..., a_n) \text{ is defined.} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Since $\chi$ is injective, this is well-defined and $\hat{\chi}$ is a weak embedding by construction. In particular, because $\Psi_{\mathcal{K}_1}$ is invariant under the renaming of constants, all terms in $\Psi_{\mathcal{K}_1}(\mathcal{B}|_{\Pi_1})$ are defined (in $(\mathcal{B}, B)$). Since $\hat{\mathcal{B}}$ and $\mathcal{B}$ are the same as $\Sigma_2$-structures we trivially have $\hat{\mathcal{B}} \models \mathcal{T}_2$, so the only thing left to show is $\hat{\mathcal{B}} \models_{\mathrm{w}} \mathcal{T}_1$. Let $D \in \mathcal{K}_1$ and let $\beta : X \to B$. We may assume that all terms $t$ of $D$ are defined in $(\hat{\mathcal{B}}, \beta)$ (otherwise there is nothing to show). We construct a valuation for $\mathcal{A}$ from this. Consider an extension term $t$ in $D$. By assumption, $\mathcal{K}_1$ is flat. This means that $t$ has the form $f(x_1, ..., x_n)$ for some variables $x_i$. Because $\beta(t)$ is defined in $\mathcal{B}$, it follows that there are elements $a_{x_1}, ..., a_{x_n}$ of $\mathcal{A}$ such that $\chi(a_{x_i}) = \beta(x_i)$ and

$$\beta(t) = f_{\hat{\mathcal{B}}}(\beta(x_1), \ldots, \beta(x_n)) = \chi(f_{\mathcal{A}}(a_{x_1}, \ldots, a_{x_n})).$$

Note that since $\chi$ is injective, the choice of the element $a_x$ for a variable $x$ is unique. Hence, we may define a map $\alpha : X \to A$ with $\alpha(x) = a_x$. As all terms in $D$ are defined in $(\hat{\mathcal{B}}, \beta)$ so they are in $(\mathcal{A}, \alpha)$ and we have $(\mathcal{A}, \alpha) \models D$ as well as $\chi(\alpha(t)) = \beta(t)$ for all terms $t$ in $D$. The claim now follows from the fact that weak embeddings preserve quantifier-free formulae in which all terms are defined.

It only remains to show that all terms of $\Psi_{\mathcal{K}_1}(D(\mathcal{B}|_{\Pi_1}))$ are defined in $\mathcal{B}$. By hypothesis, all terms in $\Psi_{\mathcal{K}_1}(D(\mathcal{A}|_{\Pi_1}))$ are defined in $(\mathcal{A}, A)$. By construction,

$$\begin{aligned} D(\mathcal{B}|_{\Pi_1}) &= \{f(b_1, \ldots, b_n) \mid b_1, \ldots, b_n \in B, f_{\mathcal{B}}(b_1, \ldots, b_n) \text{ defined } \} \\ &= \{f(\chi(a_1), \ldots, \chi(a_n)) \mid a_1, \ldots, a_n \in A, f_{\mathcal{A}}(a_1, \ldots, a_n) \text{ defined } \} \\ &= \chi(D(\mathcal{A}|_{\Pi_1})) \end{aligned}$$

The fact that all terms of $\Psi_{\mathcal{K}_1}(D(\mathcal{B}|_{\Pi_1}))$ are defined in $\mathcal{B}$ follows from the fact that $\Psi$ is invariant under constant renaming, i.e.

$$\chi(\Psi_{\mathcal{K}_1}(D(\mathcal{A}|_{\Pi_1}))) = \Psi_{\mathcal{K}_1}(\chi(D(\mathcal{A}|_{\Pi_1}))) = \Psi_{\mathcal{K}_1}(D(\mathcal{B}|_{\Pi_1})),$$

and the definition of $\hat{\mathcal{B}}$.

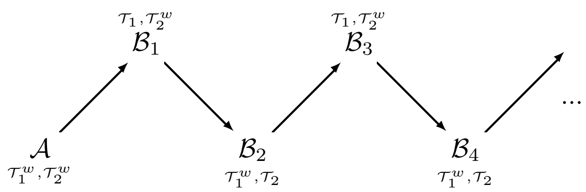
Theorem 30 lifts the results on combinations of local theories presented in [24] (cf. also [23]) to combinations of theories satisfying suitable $\Psi$-locality conditions.

**Theorem 30.** *Let $\mathcal{T}_0$ be a theory in the signature $\Pi_0$, $\Sigma_1$ and $\Sigma_2$ two disjoint sets of fresh function symbols. Let $\Pi_i$ defined as above, and let $\mathcal{K}_i$ a set of universally closed $\Pi_i$-clauses for $i = 1, 2$. Let $\mathcal{T}_i := \mathcal{T}_0 \cup \mathcal{K}_i$, $i = 1, 2$. Let $\Psi_i$ be term closure operators on ground $(\Pi_i^C)$-terms, $i = 1, 2$. Suppose that*

*(1) $\mathcal{T}_0$ is a $\forall\exists$ theory,*
*(2) $\mathcal{K}_i$ is $\Sigma_i$-flat and $\mathcal{T}_0 \subseteq \mathcal{T}_i$ satisfies condition $(\mathsf{Emb}_w^{\Psi_i})$ for $i = 1, 2$,*
*(3) all variables are shielded in $\mathcal{K}_i$, i.e., all variables occur below an extension function, $i = 1, 2$.*

*Then $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}_1 \cup \mathcal{K}_2$ has $(\mathsf{Emb}_w^{\Psi_1 \cup \Psi_2})$ where $(\Psi_1 \cup \Psi_2)(\Gamma) := \Psi_1(\Gamma) \cup \Psi_2(\Gamma)$.*

*Proof*: For notational simplicity, we drop the subscript $\mathcal{K}$ for the term operators $\Psi_i$ in this proof and say "$\Psi_i(\mathcal{A})$ is defined" to mean "$\Psi_i(\mathcal{A}|_{\Pi_i})$ is defined in $\mathcal{A}|_{\Pi_i}$". Let $\mathcal{A}$ be a partial model of $\mathcal{T}_0 \cup \mathcal{K}_1 \cup \mathcal{K}_2$ with total $\Sigma_0$-functions such that the terms in $(\Psi_1 \cup \Psi_2)(\mathcal{A})$ are all defined. We need to embed $\mathcal{A}$ into a total model of $\mathcal{T}_0 \cup \mathcal{K}_1 \cup \mathcal{K}_2$. We construct this total model inductively by repeatingly using Lemma 29 and embeddability. We obtain a diagram




where all the arrows are weak $(\Pi_1 \cup \Pi_2)$-inclusions, $\mathcal{B}_{2k} \models \mathcal{T}_2$, $\mathcal{B}_{2k} \models_w \mathcal{T}_1$ and $\mathcal{B}_{2k+1} \models \mathcal{T}_1$, $\mathcal{B}_{2k+1} \models_w \mathcal{T}_2$ as follows:

We begin by using $(\mathsf{Emb}_w^{\psi_1})$ to obtain a total model $\mathcal{C}_1$ of $\mathcal{T}_1$ into which $\mathcal{A}|_{\Pi_1}$ weakly embeds (and $\Psi_1(\mathcal{A})$ is defined). Here and hereafter we may assume w.l.o.g. that the (weak) embedding is the identity. We then use Lemma 29 to extend $\mathcal{C}_1$ to a partial model $\mathcal{B}_1$ of $\mathcal{T}_2$. We may do so because $\Psi_2(\mathcal{A})$ is defined. It also follows from Lemma 29 that $\Psi_2(\mathcal{B}_1)$ is defined. Hence, we may (weakly) embed $\mathcal{B}_1$ into a total model $\mathcal{C}_2$ of $\mathcal{T}_2$. Trivially, all terms of $\Psi_1(\mathcal{B}_1)$ are defined. We may therefore extend $\mathcal{C}_2$ to a partial model $\mathcal{B}_2$ of $\mathcal{T}_1$ (which is still a model of $\mathcal{T}_2$) in which all terms of $\Psi_1(\mathcal{B}_2)$ are defined.

Continuing in this manner we construct a chain of partial models. Now consider its union $\mathcal{B}_\omega$. $\mathcal{B}_\omega$ is a model of $\mathcal{T}_0$ because $\mathcal{T}_0$ is a $\forall\exists$ theory. Now note that

$\mathcal{B}_\omega$ is in fact a total structure. Indeed, if there exist $\bar{b}$ and $f$ (say $f \in \Sigma_1$) such that $f_{\mathcal{B}_\omega}(\bar{b})$ is undefined, then we can choose $i$ big enough such that $\bar{b} \in B_i$. Now, $i$ cannot be odd because $B_{2k+1}$ has total $\Sigma_1$-functions. Hence, $i$ is even. For the same reason, $f_{\mathcal{B}_{i+1}}(\bar{b})$ will be defined. But then it will remain defined thereafter. This is a contradiction. Analogously it can be proved that all functions in $\Sigma_2$ are defined. So $\mathcal{B}_\omega$ is a total structure.

We claim that $\mathcal{B}_\omega$ is also a (total) model of $\mathcal{T}_1 \cup \mathcal{T}_2$. We here only consider $\mathcal{K}_1$, the other case is exactly the same. Fix a valuation $\beta : X \to B_\omega$ and a clause $D \in \mathcal{K}_1$ in the variables $\bar{x}$. Let $\bar{b} := \beta(\bar{x})$. Now, choose $k$ big enough such that $\bar{b} \in \mathcal{B}_{2k+1}$. We have $\mathcal{B}_{2k+1} \models D[\bar{b}]$. And because $\mathcal{B}_{2k+1}$ is a substructure of $\mathcal{B}_\omega$ (both seen as $\Pi_1$-structures) and quantifier-free formulae are preserved under embeddings we get $\mathcal{B}_\omega \models D[\bar{b}]$ as desired. □

**Corollary 31.** *With the above notation, additionally assume that the $\mathcal{K}_i$ are $\Sigma_i$-flat and $\Sigma_i$-linear, for $i = 1, 2$. Assume that $\Psi_1$ and $\Psi_2$ are closure operators with the property that for every set $T$ of flat and linear terms, $\Psi_i(T)$ consists only of flat and linear terms. Then for any closure term operator $\Psi_3 \supseteq (\Psi_1 \cup \Psi_2)$ which has the same property it holds that $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}_1 \cup \mathcal{K}_2$ is a $\Psi_3$-local extension. Hence, under conditions (1),(3) in Theorem 30 if $\mathcal{K}_i$ are flat and linear and $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}_i$ satisfies $(\mathsf{Loc}^{\Psi_i})$, for $i = 1, 2$ then $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}_1 \cup \mathcal{K}_2$ satisfies $(\mathsf{Loc}^{\Psi_3})$ for every $\Psi_3$ with $\Psi_3 \supseteq (\Psi_1 \cup \Psi_2)$.*

*Proof.* Assume that all the assumptions of the corollary hold and $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}_i$ satisfies $(\mathsf{Loc}^{\Psi_i})$, for $i = 1, 2$. By Theorem 12, $\Psi$-locality implies $\Psi$-embeddability, so $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}_i$ satisfies $(\mathsf{Emb}_w^{\Psi_i})$, for $i = 1, 2$. By Theorem 30, $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}_1 \cup \mathcal{K}_2$ satisfies $(\mathsf{Emb}_w^{\Psi_3})$, hence, by Theorem 14, it satisfies $(\mathsf{Loc}^{\Psi_3})$.

*Example 32.* This result allows us to obtain numerous examples of local theory extensions, for instance it shows the locality of any combinations of:

- formulae in the array property fragment for a family $\Sigma_1$ of array symbols
- strict monotonicity axioms for a family $\Sigma_2$ of different array symbols (under the condition that the element theories are ordered and under the assumption that the element theories have an infinite number of elements or are e.g. the theory of total orderings, and that locality for such extensions can be proved)
- injectivity axioms for a family $\Sigma_3$ of different array symbols (under the assumption that the element theories have an infinite number of elements or are e.g. the theory of equality; i.e. such that locality for such extensions can be proved)

where $\Sigma_1, \Sigma_2, \Sigma_3$ are mutually disjoint sets.

We now analyze the locality of combinations of local extensions of a base theory which share some of the extension clauses.

**Theorem 33.** *Let $\mathcal{T}_0$ be a theory with signature $\Pi_0 = (\Sigma_0, \mathsf{Pred})$. Let $\mathcal{K}$ be a set of universally closed, flat and linear clauses in the signature $(\Sigma_0 \cup \Sigma, \mathsf{Pred})$; and let $\Sigma_1, \Sigma_2$ be sets of new function symbols such that $\Sigma_1 \cap \Sigma_2 = \emptyset$ and*

*$\Sigma \cap \Sigma_i = \emptyset$, $i = 1, 2$. Let $\mathcal{K}_i, i = 1, 2$, be sets of universally closed, flat and linear $(\Sigma_0 \cup \Sigma \cup \Sigma_i, \mathsf{Pred})$-clauses such that each clause in $\mathcal{K}_i$ contains at least one $\Sigma_i$-symbol. Assume that $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K} \cup \mathcal{K}_i$ is local, for $i = 1, 2$. Then:*

*(i) The extension $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}$ is local.*
*(ii) The extension $\mathcal{T}_0 \cup \mathcal{K} \subseteq \mathcal{T}_0 \cup \mathcal{K} \cup \mathcal{K}_i$ is local, $i = 1, 2$.*
*(iii) If $\mathcal{T}_0$ is a $\forall\exists$-theory and all variables are shielded in $\mathcal{K}_i$, $i = 1, 2$, then the extension $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K} \cup \mathcal{K}_1 \cup \mathcal{K}_2$ is local.*
*(iv) If $\mathcal{T}_0$ is a $\forall\exists$-theory and all variables are shielded in $\mathcal{K}_i$, $i = 1, 2$, then the extension $\mathcal{T}_0 \cup \mathcal{K} \cup \mathcal{K}_1 \subseteq \mathcal{T}_0 \cup \mathcal{K} \cup \mathcal{K}_1 \cup \mathcal{K}_2$ is local.*

*Proof*: (i) Let $G$ be a set of ground $(\Pi_0 \cup \Sigma \cup C)$-clauses. If $\mathcal{T}_0 \cup \mathcal{K} \cup G \models\perp$ then $\mathcal{T}_0 \cup (\mathcal{K} \cup \mathcal{K}_i) \cup G \models\perp$. Note that $\mathcal{T}_0 \cup (\mathcal{K} \cup \mathcal{K}_i) \cup G \models\perp$ if and only if $\mathcal{T}_0 \cup (\mathcal{K} \cup \mathcal{K}_i)[G] \cup G \models\perp$ if and only if $\mathcal{T}_0 \cup \mathcal{K}[G] \cup G \models\perp$. The last condition implies $\mathcal{T}_0 \cup \mathcal{K} \cup G \models\perp$. Thus, we showed that $\mathcal{T}_0 \cup \mathcal{K} \cup G \models\perp$ if and only if $\mathcal{T}_0 \cup \mathcal{K}[G] \cup G \models\perp$.

(ii) Let $\mathcal{A} \in \mathsf{PMod}_w(\Sigma_i, \mathcal{T}_0 \cup \mathcal{K} \cup \mathcal{K}_i)$. Then $\mathcal{A} \in \mathsf{PMod}_w(\Sigma \cup \Sigma_i, \mathcal{T}_0 \cup \mathcal{K} \cup \mathcal{K}_i)$, hence it weakly embeds into a total model $\mathcal{B}$ of $\mathcal{T}_0 \cup \mathcal{K} \cup \mathcal{K}_i$.

(iii) We use Theorem 30. By (ii), $\mathcal{T}_0 \cup \mathcal{K} \subseteq \mathcal{T}_0 \cup \mathcal{K} \cup \mathcal{K}_i$ is local, for $i = 1, 2$, so by Theorem 30 $\mathcal{T}_0 \cup \mathcal{K} \subseteq \mathcal{T}_0 \cup \mathcal{K} \cup \mathcal{K}_1 \cup \mathcal{K}_2$ is local. To prove that $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K} \cup \mathcal{K}_1 \cup \mathcal{K}_2$ is local, let $G$ be a set of ground clauses in the signature $(\Sigma_0 \cup \Sigma \cup \Sigma_i \cup C, \mathsf{Pred})$ such that $\mathcal{T}_0 \cup \mathcal{K} \cup \mathcal{K}_1 \cup \mathcal{K}_2, G \models \perp$. Since all sets of function symbols are pairwise disjoint, we may assume that $G$ is purified.

By locality, we obtain $\mathcal{T}_0 \cup \mathcal{K} \cup (\mathcal{K}_1 \cup \mathcal{K}_2)[G] \models \perp$. Because $\mathcal{K}_1$ and $\mathcal{K}_2$ both shield all variables, $(\mathcal{K}_1 \cup \mathcal{K}_2)[G]$ is ground. We may therefore carry out a hierarchical reduction by replacing all functions of $\Sigma_i$, $i = 1, 2$ in $(\mathcal{K}_1 \cup \mathcal{K}_2)[G]$ by their respective congruence axioms which do not contain any function symbols in $\Sigma_1 \cup \Sigma_2$. This reduction preserves (un-)satisfiability. Hence, we have $\mathcal{T}_0 \cup \mathcal{K} \cup \Gamma_0 \cup \mathsf{Con}_1 \cup \mathsf{Con}_2 \models \perp$, where $\Gamma_0$ is the purified version of $(\mathcal{K}_1 \cup \mathcal{K}_2)[G]$. The claim now follows from (i).

(iv) As in (ii). □

### 6.4 Implementation

All these results were established with the goal of having simpler, modular ways of recognizing locality and $\Psi$-locality and for giving and implementing efficient decision procedures for theory extensions and combinations. At the moment H-PILoT [13] handles combinations of local theories as follows: By Theorems 30 and 33, if a theory extension $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}_1 \cup \dots \cup \mathcal{K}_n$ is local (where the clauses $\mathcal{K}_i$ specify function symbols in mutually disjoint signatures $\Sigma_i$ and where all variables appear below an extension function) then it can equivalently be considered as a chain of local extensions $\mathcal{T}_0 \subseteq \mathcal{T}_1 = \mathcal{T}_0 \cup \mathcal{K}_1 \subseteq \dots \subseteq \mathcal{T}_n = \mathcal{T}_0 \cup \bigcup_{i=1}^n \mathcal{K}_i$, (where each extension $\mathcal{T}_i = \mathcal{T}_0 \cup \bigcup_{j=1}^i \mathcal{K}_j \subseteq \mathcal{T}_{i+1} = \mathcal{T}_i \cup \mathcal{K}_{i+1}$ is local). H-PILoT carries out a hierarchical reduction to $\mathcal{T}_0$ step-by-step if the user specifies different levels for the extension functions and it also checks that the variables are shielded at each step. Thus, the problem of checking the satisfiability of a set of ground clauses w.r.t. the extended theory is reduced automatically (in one or more steps) to a deduction problem w.r.t. the base theory, and a standard SMT-solver or a specialized prover (e.g. for the theory of reals) is used for testing the satisfiability of the formulae thus obtained after the reduction. If the result of the reduction is a satisfiable problem, H-PILoT is able to *generate a model* (which obviously is of great help when writing specifications).

## 7 Conclusions

In this paper we gave semantical characterizations of locality conditions parameterized by closure operators on ground terms. These operators capture in a theoretical way the type of instances of the axioms which are needed for guaranteeing completeness for ground satisfiability problems in extensions of a theory with sets of clauses. The conditions we imposed on the closure operators we consider allow us to address, within the framework of $\Psi$-locality, a large number of theories related to data structures and some theories which occur e.g. in relationship with description logics. Based on this, we identified several situations when locality results can be transfered from one theory extension to another one, some of them with a model theoretical flavor. We then studied possibilities of combining local theory extensions. The results we obtained allow us to identify in a simple and structured way an even larger number of local theory extensions interesting for applications. These theoretical results have been used for extending the H-PILoT prover.

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