Benchmark Examples - Stability of Nonlinear ODE's

Stefan Ratschan and Zhikun She *†

July 25, 2007

1 Examples

In this section, eight examples will be presented, for which we computed set Lyapunov functions using the method described in this paper. Here the target region TR is the set $\{x \in B : |x_i - \bar{x_i}| < \delta, 1 \le i \le n\}$, where B is a given box containing the equilibrium $\bar{x}$, and $\delta > 0$ is a arbitrarily given constant.

Example 1 *A simplified model of a chemical oscillator [2].*

$$\begin{cases} \dot{x}_1 = 0.5 - x_1 + x_1^2 x_2 \\ \dot{x}_2 = 0.5 - x_1^2 x_2 \end{cases}$$

The equilibrium is $(1, 0.5)$*. Let* $V(x_1, x_2) = ax_1^2 + bx_1 + cx_1x_2 + dx_2 + ex_2^2 + f$*, then* $\dot{V}(u, v) = 2ax_1^3x_2 + (c - 2e)x_1^2x_2^2 - cx_1^3x_2 + bx_1^2x_2 - dx_1x_2^2 - 2ax_1^2 - cx_1x_2 + (a - b + 0.5c)x_1 + (0.5c + e)x_2 + 0.5b + 0.5d.$

Choosing $B = [0.8, 1.2] \times [0.3, 0.7]$*,* $\delta = 0.01$ *and* $\varepsilon = 0.0001$*, we get a targetted Lyapunov function* $V(x_1, x_2) = x_1^2 + 2x_1x_2 + 38.7436111112000034x_2 - x_2^2$*.*

Example 2 *This is the well-known Van-der-pol equation:*

$$\begin{cases} \dot{x}_1 = -x_2 \\ \dot{x}_2 = x_1 - (1 - x_1^2)x_2 \end{cases}$$

*Stefan Ratschan is with the Institute of Computer Science, Czech Academy of Sciences, Prague, Czech Republic. Email: stefan.ratschan@cs.cas.cz.

†Zhikun She is with the School of Science, Beihang University, Beijing, China. Email: zkshe77@hotmail.com.

Let $V(x_1, x_2) = ax_1^2 + bx_1x_2 + cx_2^2$*, then* $\dot{V}(x_1, x_2) = (-2a - b + 2c)x_1x_2 + bx_1^2 + (-b - 2c)x_2^2 + bx_1^3x_2 + 2cx_1^2x_2^2$.

Choosing $B = [-0.8, 0.8] \times [-0.8, 0.8]$, $\delta = 0.1$ *and* $\varepsilon = 0.0001$*, we get a set Lyapunov function* $V(x_1, x_2) = 0.0533235085452500035x_1^2 - 0.0131336844108999994x_1x_2 + 0.0467566663398000029x_2^2$.

Example 3 *An example from [1]:*

$$\begin{cases} \dot{x}_1 = -x_1 + x_2 \\ \dot{x}_2 = 0.1x_1 - 2x_2 - x_1^2 - 0.1x_1^3 \end{cases}$$

Let $V(x_1, x_2) = ax_1^2 + bx_2^2$*, then* $\dot{V}(x_1, x_2) = -2ax_1^2 + (2a + 0.2b)x_1x_2 - 4bx_2^2 - 2bx_1^2x_2 - 0.2bx_1^3x_2$.

If we choose $B = [-0.8, 0.8] \times [-0.8, 0.8]$, $\delta = 0.1$ *and* $\varepsilon = 0.0001$*, we get a set Lyapunov function* $V(x_1, x_2) = 1.00000000000099988x_1^2 + 1.65129556434000025x_2^2$.

Example 4 *An example from a Chinese textbook on ODEs:*

$$\begin{cases} \dot{x} = -4x^3 + 6x^2 - 2x \\ \dot{y} = -2y \end{cases}$$

Let $V(x, y) = ax^4 + bx^3 + cx^2 + dy^2$*, then* $\dot{V}(x, y) = -16ax^6 + (24a - 12b)x^5 + (-8a + 18b - 8c)x^4 + (-6b + 12c)x^3 - 4cx^2 - 4dy^2$.

If we choose $B = [-0.4, 0.4] \times [-0.4, 0.4]$, $\delta = 0.1$ *and* $\varepsilon = 0.000001$*, we get a set Lyapunov function* $V(x, y) = 1.00000000000250000x^4 + 0.571428571430000032x^3 + 0.285714285715000016x^2 + 1.52556785714249998y^2$.

Example 5 *An example from [3] whose Lyapunov function has been constructed using the sum of squares decomposition:*

$$\begin{cases} \dot{x} = -x + (1 + x)y \\ \dot{y} = -(1 + x)x \end{cases}$$

Let $V(x, y) = ax^2 + bxy + cy^2 + dy^3 + ex^4 + fx^2y^2 + gy^4$*, then* $\dot{V}(x, y) = (-2a - b)x^2 + (2a - b - 2c)xy + by^2 - bx^3 + (2a - 2c)x^2y + (b - 3d)xy^2 + (-4e)x^4 + (4e - 2f)x^3y + (-3d - 2f)x^2y^2 + (2f - 4g)xy^3 + (4e - 2f)x^4y + (2f - 4g)x^2y^3$.

If we choose $B = [-0.7, 0.9] \times [-0.7, 0.9]$, $\delta = 0.1$ *and* $\varepsilon = 0.0001$*, we get a set Lyapunov function* $V(x, y) = 1.00719424460431672x^2 - 0.0143884892086333319xy + 1.01438848920861679y^2 - 0.00479616306954444398y^3 - 0.0107913669064749994x^4 + 0.0215827338129499988x^2y^2 + 0.0107913669064749994y^4$.

Example 6 *A three-dimensional example from [5]:*

$$\begin{cases} \dot{x}_1 = -x_2 \\ \dot{x}_2 = -x_3 \\ \dot{x}_3 = -x_1 - 2x_2 - x_3 + x_1^3 \end{cases}$$

Let $V(x_1, x_2, x_3) = ax_1^2 + bx_2^2 + cx_3^2 + dx_1x_2 + ex_1x_3 + fx_2x_3$, *then* $\dot{V}(x_1, x_2, x_3) = (-d + e)x_1^2 - 2fx_2^2 + (-2c - f)x_3^2 + (-2a - 2e - f)x_1x_2 + (-2c - d - e)x_1x_3 + (-2b - e - f)x_2x_3 + 2cx_1^3x_3 + ex_1^4 + fx_1^3x_2$.

If we choose $B = [-0.2, 0.2] \times [-0.2, 0.2] \times [-0.2, 0.2]$, $\delta = 0.1$ *and* $\varepsilon = 0.0001$, *we get a set Lyapunov function* $V(x_1, x_2, x_3) = 1.23090731070160020x_1^2 + 0.59742167101660042x_2^2 + 0.76697127937499981x_3^2 - 0.26697127937999870x_1x_2 - 1.26697127937000009x_1x_3 + 0.07212793733679999937x_2x_3$.

Example 7 *An example from a Chinese textbook on ODEs:*

$$\begin{cases} \dot{x} = -x - 3y + 2z + yz \\ \dot{y} = 3x - y - z + xz \\ \dot{z} = -2x + y - z + xy \end{cases}$$

Let $V(x, y, z) = ax^2 + by^2 + cz^2$, *then* $\dot{V}(x, y, z) = -2ax^2 - 2by^2 - 2cz^2 + (-6a + 6b)xy + (4a - 4c)xz + (-2b + 2c)yz + (2a + 2b + 2c)xyz$

If we choose $B = [-0.4, 0.4] \times [-0.4, 0.4] \times [-0.4, 0.4]$, $\delta = 0.1$ *and* $\varepsilon = 0.0001$, *we get a set Lyapunov function* $V(x, y, z) = x^2 + y^2 + z^2$.

Example 8 *A six-dimensional system from [2]:*

$$\begin{cases} \dot{x}_1 = -x_1^3 + 4x_2^3 - 6x_3x_4 \\ \dot{x}_2 = -x_1 - x_2 + x_5^3 \\ \dot{x}_3 = x_1x_4 - x_3 + x_4x_6 \\ \dot{x}_4 = x_1x_3 + x_3x_6 - x_4^3 \\ \dot{x}_5 = -2x_2^3 - x_5 + x_6 \\ \dot{x}_6 = -3x_3x_4 - x_5^3 - x_6 \end{cases}$$

Let $V(x_1, x_2, x_3, x_4, x_5, x_6) = ax_1^2 + bx_2^4 + cx_3^2 + dx_4^2 + ex_5^4 + fx_6^2$, *then* $\dot{V}(x_1, x_2, x_3, x_4, x_5, x_6) = -2ax_1^4 - 4bx_2^4 - 2cx_3^4 - 2dx_4^4 - 4ex_5^2 - 2fx_6^2 + (8a - 4b)x_1x_2^3 + (-12a + 2c + 2d)x_1x_3x_4 + (4b - 8e)x_2^3x_5^3 + (2c + 2d - 6f)x_3x_4x_6 + (4e - 2f)x_5^3x_6$.

Choosing $B = [-0.8, 0.8] \times \cdots \times [-0.8, 0.8]$, $\delta = 0.1$ *and* $\varepsilon = 0.0001$, *we can get a set Lyapunov function* $V(x_1, x_2, x_3, x_4, x_5, x_6) = x_1^2 + 2x_2^4 + 5.5x_3^2 + 0.5x_4^2 + x_5^4 + 2x_6^2$.

References

[1] R. Genesio, M. Tartaglia, and A. Vicino. On the estimation of asympototic stability regions: state of the art and new proposals. *IEEE Trans. on Automatic Control*, 30(8):747–755, 1985.

[2] A. Papachristodoulou and S. Prajna. On the construction of Lyapunov functions using the sum of squares decomposition. In *Proc. of the IEEE Conf. on Decision and Control*, 2002.

[3] P. Parrilo and S. Lall. Semidefinite programming relaxations and algebraic optimization in control. *European Journal of Control*, 9(2–3), 2003.

[4] S. Ratschan and Z. She. Providing a basin of attraction to a target region by computation of Lyapunov-like functions. In *IEEE Int. Conf. on Computational Cybernetics*, 2006.

[5] D. N. Shields and C. Storey. The behaviour of optimal Lyapunov functions. *Int. J. Control*, 21(4):561–573, 1975.