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by
Viorica Sofronie-Stokkermans

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Author(s) contact: Viorica Sofronie-Stokkermans (vs@avacs.org).

Sheaves and geometric logic and applications to modular verification of complex systems^{*}

Viorica Sofronie-Stokkermans

Max-Planck-Institut für Informatik, Campus E 1.4, Saarbrücken, Germany
e-mail: sofronie@mpi-inf.mpg.de

Abstract. In this paper we show that states, transitions and behavior of concurrent systems can often be modeled as sheaves over a suitable topological space (where the topology expresses how the interacting systems share the information). This allows us to use results from categorical logic (and in particular geometric logic) to describe which type of properties are transferred, if valid locally in all component systems, also at a global level, to the system obtained by interconnecting the individual systems. The main area of application is to modular verification of complex systems. We illustrate the ideas by means of an example involving a family of interacting controllers for trains on a rail track.¹

1 Introduction

Complex systems, consisting of several components that interact, arise in a natural way in a wide range of applications. The components may be complex systems themselves or may be simple - but even then their composition can be complicated because of the necessity to take into account the interaction between the single components. One of the problems that arise in the verification of such complex systems is that the state space can grow exponentially with the number of components. Symbolic representations of states and symbolic model checking have greatly increased the size of the systems that can be verified. However, many realistic systems are still too large to be handled. It is therefore important to find techniques that can be used to further extend the size of the systems that can be verified. One possibility is to check properties in a modular way (i.e. verify them for the individual components, infer that they also hold in the system obtained by the interconnection of the individual components, and then use them to deduce additional properties of the system). Not all properties of individual systems are preserved when interconnecting them: for instance deadlocks might occur when interconnecting deadlock free systems.

In this paper we address the following question, very important in verification:

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¹ This is the extended version of a paper with the same name which has been accepted for publication in the ENTCS proceedings of GETCO 2006.

Which properties of complex systems can be checked in a modular way?

To answer this question, we here use an analogy with phenomena in topology and algebraic geometry, where “locally defined” objects are studied, which need to be “patched” to “globally defined” objects. An important tool used in algebraic geometry in the study of such phenomena is sheaf theory.

Informally, a sheaf over a topological space is a map F which associates with every open set U of the topological space a set $F(U)$ of *locally defined data* attached to the open set U . The data can be restricted to smaller open sets, and for every open set U , any collection of compatible data assigned to a family of smaller open sets $\{U_i \mid i \in I\}$ covering the original set U can be “glued together” to an element in $F(U)$.

We show that also in the study of complex systems sheaf theory allows us to establish links between “local” and “global” description of systems as well as between “local” and “global” properties of systems. In our setting, the open sets model the component collections of subsystems; the topological space we use expresses how the interacting systems share the information. The *data* assigned to the open sets are, in our approach, descriptions of the *states, possible parallel actions, transitions*, or of the *behavior* of the systems, or of the local time of the systems. We show that, given a family of interacting systems, states, actions, transitions, behavior in time have the property that any collection of compatible data (i.e. “local” states, actions, transitions, resp. behavior) assigned to a family of interacting systems can be “glued together” to a “global” element (state, action, transition, or behavior). In other words, we show that states, actions, transitions, behavior in time can often be modeled by sheaves over this topological space which describes how the interacting systems share the information.

Many properties of systems can be expressed as assertions about states, actions, transitions, behavior in time. The sheaf semantics allows us to prove, by using results from geometric logic, that those properties of systems that can be expressed by *cartesian axioms* are preserved after interconnecting the systems.

The starting point of our research is the work of Goguen [6], who uses sheaves to model behavior in an ‘interval of observation’, and the work of Monteiro and Pereira [13], who model behavior using sheaves of monoids. The idea of modeling states, actions and transitions by sheaves with respect to a topological space, and of using geometric logic for studying the link between properties of the components and properties of the systems that arises from their interconnection occurs, to the best of our knowledge, for the first time in our previous work [16,17,18]. We present an overview of our results in [17,18] together with new results which illustrate how sheaf theory can be used for the modular verification of complex systems. We illustrate all the notions introduced by means of a running example involving a family of interacting controllers controlling a subsets of consecutive trains on a linear, loop-free rail track. The main contributions of the paper are summarized below:

- We start with a presentation of our results previously described also in [16,17,18], where we showed that states, parallel actions, transitions and behavior in time can be modeled by sheaves. In order to make the paper self-contained we provide all the proofs. In addition, we illustrate the various notions we use (definition of systems, states, parallel actions, transitions, conditions on transition relations, categorical constructions, covers, gluing and sheaf properties) by means of a running example referring to a system of trains.
- In addition to the model of behavior we considered in [16,17,18], we also analyze a description of behavior by traces of execution (modeled by free monoids and partially commutative monoids). We analyze gluing and sheaf properties in this context. We pay special attention to identifying situations when the stalks of the sheaves are isomorphic to the behavior of the individual systems, whereas the global sections are isomorphic to the behavior of the colimit of these systems. For this, we use results on sheaf representation in universal algebra. We establish links with existing results in the study of Petri nets and Mazurkiewicz traces [3] and on modeling behavior by sheaves of monoids [13].
- We use geometric logic for describing properties which can be checked modularly. We illustrate the ideas on the running example, and give examples of systems (describing trains on a linear, acyclic rail track) for which safety properties as well as certain liveness properties can be checked in a modular way.

Structure of the paper. The paper is structured as follows. In Section 2 we present a model for *systems* and for their *states, parallel actions and transitions*. A category of systems is introduced and pullbacks and colimits in this category are described. In Section 3 we introduce the notions of presheaves, sheaves and bundles and the relationships between them, and present some results on sheaves of algebras. In Section 4 we give a model for complex, interacting systems, and motivate the use of sheaf theory. Sections 5 to 8 describe our sheaf-theoretic semantics for states, parallel actions, transitions and behavior. In Section 9 geometric logic is used to test preservation of 'local' properties under connection of systems. Several examples are given in Section 10.

2 Systems

Our aim is to model interconnected systems. In this section we will give a definition for systems and their states, (parallel) actions, and transitions. We will then present our model of communication between systems (using common subsystems through which information is exchanged) and introduce a category of systems.

2.1 Systems: Definition

We assume that systems are described by:

- a set X of control variables of the system, a set Γ of constraints on X expressed in a language \mathcal{L} , describing dependencies between control variables.
- a set A of atomic actions, and a set C of Boolean constraints on A describing dependencies between atomic actions (e.g. specifying which actions *must* be performed at the same time, and which *are not allowed* to be performed at the same time).

Let $\Sigma = (\text{Sort}, O, P)$ be a signature, consisting of a set Sort of sorts, a set O of operation symbols and a set P of predicate symbols. For a (many-sorted) set of variables $X = \{X_s\}_{s \in \text{Sort}}$ let $\text{Fma}_\Sigma(X)$ be the set of formulae over Σ .

A Σ -structure is a structure $M = ((M_s)_{s \in \text{Sort}}, \{f_M\}_{f \in O}, \{R_M\}_{R \in P})$ where if $f \in O$ has arity $s_1 \dots s_n \rightarrow s$ then $f_M : M_{s_1} \times \dots \times M_{s_n} \rightarrow M_s$ and if $R \in P$ has arity $s_1 \dots s_n$ then $R_M \subseteq M_{s_1} \times \dots \times M_{s_n}$. The class of all Σ -structures is denoted Str_Σ .

If $M \in \text{Str}_\Sigma$, $s : X \rightarrow M$ is a sort-preserving assignment, and $\phi \in \text{Fma}_\Sigma(X)$, $(M, s) \models \phi$ (abbreviated by $s \models \phi$) is defined in the usual way (cf. [1], Ch. 1). We can define a relation \models_M by: $\Gamma \models_M \phi$ iff for every assignment $s : X \rightarrow M$ of values in M to the variables in X , if $s \models \gamma$ for every $\gamma \in \Gamma$, then $s \models \phi$.

Definition 1 A system S is a tuple $(\Sigma, X, \Gamma, M, A, C)$, where

- (i) $\Sigma = (\text{Sort}, O, P)$ and $X = \{X_s\}_{s \in \text{Sort}}$ are as specified above; together they define the language \mathcal{L}_S of the system S ;
- (ii) $\Gamma \subseteq \text{Fma}_\Sigma(X)$ is a set of constraints, which is closed with respect to the semantical consequence relation \models_M ;
- (iii) $M \in \text{Str}_\Sigma$;
- (iv) A is a set of actions; for every $a \in A$, a set $X^a \subseteq X$ of variables on which a depends, and a transition relation $Tr^a \subseteq St^a \times St^a$, where $St^a = \{s|_{X^a} \mid s : X \rightarrow M, s \models \Gamma\}$ are specified;
- (v) C is a set of constraints on actions, expressed by boolean equations over $F_B(A)$ (the free boolean algebra generated by A) stating e.g. which actions can (or have to) be executed in parallel, and which cannot; C must contain all boolean equations that can be deduced from C .

In what follows, we may refer to any of the components of a system S by adding S as a subscript, e.g. Σ_S for its signature. X_S^a will denote the minimal set of variables on which $a \in A_S$ depends, and Tr_S^a the transition relation associated with a .

For the sake of simplicity, in the examples below we will only mention explicitly the axioms in Γ and C and not all their consequences.

Example 1 We consider a system consisting of n consecutive trains on a linear track controlled by a radio controller (cf. also [8]), see also Fig. 1. The trains report their position to the controller at fixed time intervals Δt . The controller analyzes the distances between successive trains (we assume that certain security distance treshholds $l_0 < l_1 < \dots < l_m < \dots$ and corresponding maximal speed

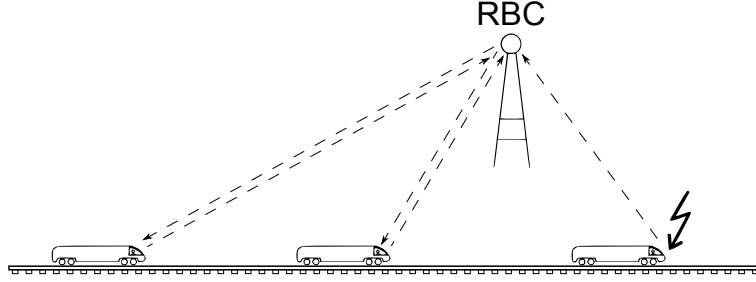


Fig. 1. Example: Several trains controlled by RBC

limits $\text{maxSpeed}(1) < \dots < \text{maxSpeed}(m) < \dots$, deemed to be safe for the trains, are known) and updates the movement modes of trains accordingly. A train with movement mode k can move in the next time interval Δt with an arbitrary speed between a minimal speed and the maximal speed limit of mode k , $\text{maxSpeed}(k)$. The system is modeled as follows:

- (i) *Language*: $\Sigma = (\text{Sort}, O, P)$, where $\text{Sort} = \{\text{real}, \text{nat}\}$;
 - $O = \{+, -, \text{minSpeed}, \text{maxSpeed}, \text{succ}\}$, where $+, - : \text{real}, \text{real} \rightarrow \text{real}$, minSpeed is a constant of sort real , maxSpeed a function of arity $\text{nat} \rightarrow \text{real}$, and succ of arity $\text{nat} \rightarrow \text{nat}$.
 - $P = \{\leq\}$, where \leq has arity real, real .
 - $X = \bigcup_{i=1}^n \{\text{TrainIndex}_i, \text{ActualPos}_i, \text{RepPos}_i, \text{Mode}_i\}$, where TrainIndex_i controls the number of train i on the line track, and $\text{ActualPos}_i, \text{RepPos}_i$ and Mode_i control the actual, resp. reported position and the movement mode of train i respectively.
- (ii) *Constraints*: $\Gamma = \{\text{succ}(\text{TrainIndex}_i) = \text{TrainIndex}_{i+1} \mid i \in \{1, \dots, n-1\}\}$.
- (iii) *Model* $M = (M_{\text{nat}}, M_{\text{real}}, +, -, \text{minSpeed}, \text{maxSpeed}, \text{succ}, \leq)$, where $M_{\text{nat}} = \mathbb{N}$; $M_{\text{real}} = \mathbb{R}$; $+, -$ are addition and subtraction on \mathbb{R} , $\text{succ} : \mathbb{N} \rightarrow \mathbb{N}$ is the successor function, $\text{minSpeed} \in \mathbb{R}$, $\text{maxSpeed} : \mathbb{N} \rightarrow \mathbb{R}$ associates with a mode $k \in \mathbb{N}$ the maximal allowed speed in mode k ; \leq is the order relation on \mathbb{R} .
- (iv) *Actions*: $A = \{\text{report}_i \mid i \in \{1, \dots, n\}\} \cup \{\text{update}\} \cup \{\text{move}_i \mid i \in \{1, \dots, n\}\}$.
 - report_i depends on the variables $X^{r_i} = \{\text{ActualPos}_i, \text{RepPos}_i, \text{Mode}_i\}$.
If $s, s' : X \rightarrow M$ then $(s|_{X^{r_i}}, s'|_{X^{r_i}}) \in Tr^{r_i}$ iff the following three conditions are fulfilled:
 - (i) $s(\text{Mode}_i) = 0$
 - (ii) $s'(\text{RepPos}_i) = s(\text{ActualPos}_i)$
 - (iii) $s'(\text{ActualPos}_i) = s(\text{ActualPos}_i)$.

- update depends on $X^u = \bigcup_{i \in \{1, \dots, n\}} \{\text{ActualPos}_i, \text{RepPos}_i, \text{Mode}_i\}$.
 If $s, s' : X \rightarrow M$ then $(s|_{X^u}, s'|_{X^u}) \in Tr^{r_i}$ if and only if for all $i \in \{1, \dots, n\}$ the following hold:
 - (i) $s(\text{Mode}_i) = 0$,
 - (ii) $s'(\text{ActualPos}_i) = s(\text{ActualPos}_i)$,
 - (iii) $s'(\text{RepPos}_i) = s(\text{ActualPos}_i)$, and
 - (iv) $s'(\text{Mode}_i)$ is updated according to the following rules:
 $s'(\text{Mode}_1) > 0$ and for all $i \geq 2$:
 if $l_k < s(\text{RepPos}_{i-1}) - s(\text{RepPos}_i) \leq l_{k+1}$ then $s'(\text{Mode}_i) = k + 1$.
- move_{*i*} depends on $X^{m_i} = \{\text{ActualPos}_i, \text{Mode}_i\}$.
 - It is enabled at a state s iff $s(\text{Mode}_i) > 0$ for all $i \in \{1, \dots, n\}$;
 - it changes ActualPos_i according to the value of Mode_i as follows, for $i \in \{1, \dots, n\}$ $s'(\text{ActualPos}_i) \in [\text{PosMin}, \text{PosMax}]$ where:
 $\text{PosMin} = \text{RepPos}_i + \Delta t * \text{minSpeed}$
 $\text{PosMax} = \text{RepPos}_i + \Delta t * \text{maxSpeed}(s(\text{Mode}_i))$;
 - It updates the value of Mode_i to 0: $s'(\text{Mode}_i) = 0$ for $i \in \{1, \dots, n\}$.
- (v) *Constraints on actions*: $C = \{\text{report}_1 = \text{report}_2 = \dots = \text{report}_n = \text{update}\} \cup \{\text{report}_i \wedge \text{move}_i = 0 \mid i \in \{1, \dots, n\}\} \cup \{\text{move}_1 = \dots = \text{move}_n\}$.

2.2 States, parallel actions

It is important to describe *the states* of a system and *the actions which can be performed in parallel* (which we here name admissible parallel actions).

Definition 2 Let $S = (\Sigma, X, \Gamma, M, A, C)$ be a system.

- A state of S is an assignment $s : X \rightarrow M$ satisfying all formulae in Γ . The set of states of the system S is $St(S) = \{s : X \rightarrow M \mid s \models \Gamma\}$.
- The admissible parallel actions of S are sets of actions, represented by maps $f : A \rightarrow \{0, 1\}$ that satisfy all constraints in C . The set of admissible parallel actions of S is the set $Pa(S) = \{f : A \rightarrow \{0, 1\} \mid f \text{ satisfies } C\}$.

Below we restrict our attention to *finite* systems, i.e. systems whose signatures, sets of control variables and sets of actions are finite; this suffices for practical applications and avoids having to consider infinitely many actions occurring in parallel.

Example 2 Consider the system S in Example 1 with $n \geq 2$. A state is a map $s : X \rightarrow M$ which satisfies Γ . For example, any $s : X \rightarrow M$ such that:

- $s(\text{TrainIndex}_1) = 1, s(\text{TrainIndex}_2) = 2, \dots, s(\text{TrainIndex}_n) = n$ or
- $s(\text{TrainIndex}_1) = 100, s(\text{TrainIndex}_2) = 101, \dots, s(\text{TrainIndex}_n) = 100 + (n - 1)$.

is a state of S . If $s(\text{TrainIndex}_1) = 1$ and $s(\text{TrainIndex}_2) = 3$, s cannot be a state.

An admissible parallel action is a map $f : A \rightarrow \{0, 1\}$ which satisfies the constraints in C . Examples of admissible parallel actions are

1. $f(\text{report}_1)=f(\text{report}_2)=\dots=f(\text{report}_n)=f(\text{update})=1$, and 0 otherwise,
2. $f(\text{move}_1)=\dots=f(\text{move}_n)=1$ and 0 otherwise.

Any map f with $f(\text{move}_1)=f(\text{report}_1)=1$, or with $f(\text{report}_i)=0$ but $f(\text{update})=1$, is not an admissible parallel action, since it does not satisfy the constraints in C .

2.3 Transitions

Let $S = (\Sigma, X, F, M, A, C)$ be a system. Let $Tr_S(a) = \{(s_1, s_2) \mid s_1, s_2 \in St(S), (s_1|_{X^a}, s_2|_{X^a}) \in Tr^a, s_1(x) = s_2(x) \text{ if } x \notin X^a\}$. We extend the notion of transition to parallel actions. For this we present two (non-equivalent) properties of transitions that express compatibility of the actions in an admissible parallel action:

(Disj) Let $f \in Pa(S), s \in St(S)$ such that for every $a \in f^{-1}(1)$ there is an $s^a \in St(S)$ with $(s|_{X^a}, s^a|_{X^a}) \in Tr^a$. Then for all $a, b \in f^{-1}(1)$ and $x \in X^a \cap X^b$, $s^a(x) = s^b(x)$ (the new local states agree on intersections). Then the transition associated with f is defined as:

$$Tr_S(f) = \{(s, t) \mid s, t \in St(S), (s|_{X^a}, t|_{X^a}) \in Tr^a \text{ for every } a \text{ with } f(a) = 1 \text{ and } s(x) = t(x) \text{ if } x \notin \bigcup_{a, f(a)=1} X^a\}.$$

The property **(Disj)** applies when a parallel action $f : A \rightarrow \{0, 1\}$ is admissible iff its components do not consume common resources. This happens e.g. if for all $a_1, a_2 \in A$ with $f(a_1) = f(a_2) = 1$, either $a_1 = a_2 \in C$ or X^{a_1} and X^{a_2} are disjoint. In concurrency theory, this property is called “real parallelism” or “independence”.

Example 3 Consider the example in Section 1. Let $f : A \rightarrow \{0, 1\}$ be an admissible parallel action. We have two possibilities:

- (i) $f(\text{report}_1) = \dots = f(\text{report}_n) = f(\text{update}) = 1$ and 0 otherwise.
The transition relation of this parallel action updates the value of each variable RepPos_i according to the transition relation of report_i , resp. update . The changes are not contradictory, since the effect of update agrees with the effect of $\text{report}_1, \dots, \text{report}_n$ on the variables in $X^u \cap X^{r_i}$. Thus, **(Disj)** holds.
- (ii) $f(\text{report}_1) = \dots = f(\text{report}_n) = f(\text{update}) = 0$ and $f(\text{move}_1) = \dots = f(\text{move}_n) = 1$ and f is 0 otherwise. As the actions $\text{move}_j, j = 1, \dots, n$ depend on disjoint sets of variables, **(Disj)** is satisfied also in this case.
The transition relation of this parallel action updates the value of each variable ActualPos_i . Since the sets of variables these actions depend upon, namely X^{m_i} , are mutually disjoint, these changes cannot be contradictory.

- (Indep)** Assume that if $a = b \in C$ then $X^a = X^b$ and $Tr^a = Tr^b$, and a and b can both be identified with one action, namely the parallel execution of a and b . Let $f \in Pa(S)$, $s \in St(S)$. We identify all $a, b \in A$ with $a = b \in C$ and $f(a) = f(b) = 1$. Let $\{b_1, \dots, b_m\} \subseteq f^{-1}(1)$. We assume that:
- (i) $g : A \rightarrow \{0, 1\}$, defined by $g(a)=1$ iff $a \in \{b_1, \dots, b_m\}$, is in $Pa(S)$;
 - (ii) if $s \xrightarrow{b_1} s_1 \xrightarrow{b_2} s_2 \rightarrow \dots \rightarrow s_{m-1} \xrightarrow{b_m} t$ then for every permutation σ of $\{1, \dots, m\}$, there exist states $t_1^\sigma, \dots, t_{m-1}^\sigma$ such that
$$s \xrightarrow{b_{\sigma(1)}} t_1^\sigma \xrightarrow{b_{\sigma(2)}} t_2^\sigma \rightarrow \dots \rightarrow t_{m-1}^\sigma \xrightarrow{b_{\sigma(m)}} t$$

Then the transition relation induced by f is:

$$Tr_S(f) = \{(s, t) \mid s, t \in St(S), \text{ and } \exists s_0, s_1, \dots, s_{n-1}, s_n \in St(S) \text{ s.t. } s_0 = s, s_n = t, \text{ and } (s_{i-1}, s_i) \in Tr_S(a_i), \text{ for all } 1 \leq i \leq n\}.$$

It is easy to see that if $(s, t) \in Tr_S(f)$ then $s(x)=t(x)$ for every $x \notin \bigcup_{a, f(a)=1} X^a$.

The property **(Indep)** reflects how transitions are interpreted when actions to be performed in parallel do consume common resources. It applies if the state reached after executing an action is uniquely determined: the fact that all components of a parallel action $f : A \rightarrow \{0, 1\}$ can be applied at a state s is a necessary condition for f to be applicable at state s , but in general not sufficient (in addition, one has to ensure that there are enough resources to perform all actions).

- Condition **(Indep)**(i) holds e.g. if C is the set of all consequences of a set C_0 consisting only of formulae of the form $a_1 = a_2$ and $a_1 \wedge a_2 = 0$.
- Condition **(Indep)**(ii) states that the final state does not depend on the order in which the actions are executed (it is related to the notions of interleaving and permutable actions used in concurrency).

Example 4 We consider a variant of Example 1, in which we assume that there is no control unit, but all trains have access to all information about the positions of all trains. The trains report all together and move all together. The actions are $A = \{\text{report}_1, \dots, \text{report}_n\} \cup \{\text{move}_1, \dots, \text{move}_n\}$, with constraints $C = \{\text{report}_1 = \dots = \text{report}_n\} \cup \{\text{move}_1 = \dots = \text{move}_n\} \cup \{\text{report}_i \wedge \text{move}_i = 0 \mid i \in \{1, \dots, n\}\}$.

Let $f : A \rightarrow \{0, 1\}$ be an admissible parallel action. Then $f^{-1}(1)$ is either \emptyset or $\{\text{report}_1, \dots, \text{report}_n\}$ or $\{\text{move}_1, \dots, \text{move}_n\}$. As in all cases the actions in $f^{-1}(1)$ depend on disjoint sets of variables, the final state does not depend on the order in which the actions would be performed sequentially.

2.4 A category of systems

Essential to our model for communication is that systems have common subsystems through which information exchange is made.

Definition 3 Let S, T be two systems. We say that S is a subsystem of T (denoted $S \succrightarrow T$) if $\Sigma_S \subseteq \Sigma_T$, $X_S \subseteq X_T$, $A_S \subseteq A_T$, the constraints in Γ_S (resp. C_S) are consequences of the constraints in Γ_T (resp. C_T), and $M_S = M_T|_{\Sigma_S}$ (the reduct of M_T to the signature Σ_S).

Let $S \succrightarrow T$. If we regard a transition in T from the perspective of S , some variables in S may change their values with no apparent cause, namely if some action in A_T but not in A_S is performed, which depends on variables in X_S . If this cannot be the case, we call the subsystem $S \succrightarrow T$ *transition-connected*. Formally:

Definition 4 S is a transition-connected (t.c.) subsystem of T (denoted $S \leftrightarrow T$) if $S \succrightarrow T$ and the following two conditions hold:

- (T₁) If $a \in A_T$ and $X_T^a \cap X_S \neq \emptyset$ then $a \in A_S$, and $X_S^a = X_T^a \cap X_S$.
- (T₂) If $a \in A_S$, $s_1, s_2 \in St(T)$, and $(s_1|_{X_T^a}, s_2|_{X_T^a}) \in Tr_T^a$ then $(s_1|_{X_S^a}, s_2|_{X_S^a}) \in Tr_S^a$.

It is easy to see that the relation \leftrightarrow is a partial order on systems.

Example 5 Consider the system $S = (\Sigma, X, \Gamma, M, A, C)$ in Example 1. Let $1 \leq k \leq l \leq n$ and $I = \{k, \dots, l\}$. Consider the subsystem S_k^l of S consisting of the consecutive trains controlled by the variables in $\{\text{TrainIndex}_i \mid i \in I\}$ (which can be seen as a system of trains with its own, “local” controller, cf. Fig. 2).

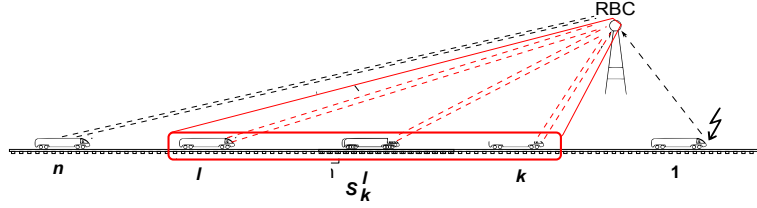


Fig. 2. Case $k \leq l$: S_k^l

Then $S_k^l = (\Sigma, X_k^l, \Gamma_k^l, M, A_k^l, C_k^l)$, where:

- $X_k^l = \bigcup_{i \in I} \{\text{TrainIndex}_i, \text{ActualPos}_i, \text{RepPos}_i, \text{Mode}_i\}$,
- $\Gamma_k^l = \{\text{succ}(\text{TrainIndex}_i) = \text{TrainIndex}_{i+1} \mid i \in \{k, \dots, l-1\}\}$,
- $A_k^l = \{\text{report}_i \mid i \in I\} \cup \{\text{update}\} \cup \{\text{move}_i \mid i \in I\}$, and
- C_k^l is the restriction of C to the actions in A_k^l : $C_k^l = \{\text{report}_i = \text{update} \mid i \in I\} \cup \{\text{report}_i \wedge \text{move}_i = 0 \mid i \in I\} \cup \{\text{move}_k = \dots = \text{move}_l\}$.

Condition (T₁) obviously holds: if an action of S depends on variables known in S_k^l , then the action is known in S_k^l . Condition (T₂) obviously holds for $\{\text{report}_i \mid i \in I\} \cup \{\text{move}_i \mid i \in I\}$ and, for update , for all trains which follow a train known in S_k^l . For the first train (T₂) is a consequence of the fact that the mode update restrictions in S are stronger than those in S_k^l (any mode allowed in S is still allowed in S_k^l).

We define a category TcSys having as objects systems, and a morphism $S \hookrightarrow T$ between S and T whenever S is a t.c. subsystem of T . TcSys has *pullbacks* (infimums with respect to this order of t.c. subsystems of a given system; we will denote this operation by \wedge) and *colimits* of diagrams of t.c. subsystems of a given system.

Proposition 5 *The category TcSys has pullbacks.*

Proof: Let $S_1 \hookrightarrow S$ and $S_2 \hookrightarrow S$, where $S = (\Sigma, X, \Gamma, M, A, C)$, $S_i = (\Sigma_i, X_i, \Gamma_i, M_i, A_i, C_i)$. Then $M_i = M|_{\Sigma_i}$, and for every $a \in A_i$, $X_i^a = X_S^a \cap X_i$ ($i = 1, 2$). Hence, for every $a \in A_1 \cap A_2$, $X_1^a \cap X_2^a = X_S^a \cap X_1 \cap X_2$.

Let $S_{12} = (\Sigma_1 \cap \Sigma_2, X_1 \cap X_2, \Gamma_1 \cap \Gamma_2, M_{S|_{\Sigma_1 \cap \Sigma_2}}, A_1 \cap A_2, C_1 \cap C_2)$, and such that for every $a \in A_1 \cap A_2$, $X_{12}^a = X_1^a \cap X_2^a = X_S^a \cap X_1 \cap X_2$, and $Tr_{12}^a = \{(s_1|_{X_{12}^a}, s_2|_{X_{12}^a}) \mid s_1, s_2 \in St(S_1), (s_1|_{X_1^a}, s_2|_{X_1^a}) \in Tr_{S_1}^a\} \cup \{(s_1|_{X_{12}^a}, s_2|_{X_{12}^a}) \mid s_1, s_2 \in St(S_2), (s_1|_{X_2^a}, s_2|_{X_2^a}) \in Tr_{S_2}^a\}$. It is easy to see that S_{12} is a transition-connected subsystem of both S_1 and S_2 , and has the universality property of a pullback. \square

Proposition 6 *Let $S = (\Sigma, X, M, \Gamma, A, C)$ be a system and $\{S_i \hookrightarrow S \mid i \in I\}$ a family of transition-connected subsystems of S , where for every $i \in I$, $S_i = (\Sigma_i, X_i, M_i, \Gamma_i, A_i, C_i)$. The colimit of this family in TcSys is the system \overline{S} with:*

- $\Sigma_{\overline{S}} = \bigcup_{i \in I} \Sigma_i$,
- $X_{\overline{S}} = \bigcup_{i \in I} X_i$,
- $M_{\overline{S}} = M|_{\bigcup_{i \in I} \Sigma_i}$,
- $\Gamma_{\overline{S}} = (\bigcup_{i \in I} \Gamma_i)^\bullet$ (the family of all logical consequences of $\bigcup_{i \in I} \Gamma_i$),
- $A_{\overline{S}} = \bigcup_{i \in I} A_i$,
- $C_{\overline{S}} = (\bigcup_{i \in I} C_i)^\bullet$ (the family of all logical consequences of $\bigcup_{i \in I} C_i$),

and where for every $a \in \bigcup_{i \in I} A_i$

- $X_{\overline{S}}^a = \bigcup_{a \in A_i} X_i^a$, and
- $Tr_{\overline{S}}^a = \{(s_1|_{X_{\overline{S}}^a}, s_2|_{X_{\overline{S}}^a}) \mid s_1, s_2 \in St(\overline{S}), \text{ and for every } i \in I \text{ with } a \in A_i, (s_1|_{X_i^a}, s_2|_{X_i^a}) \in Tr_{S_i}^a\}$.

Proof: (Sketch) One needs to show that for every $i \in I$, S_i is a transition-connected subsystem of \overline{S} , and that \overline{S} satisfies the universality property of a colimit. The proof is long, but straightforward. \square

Example 6 Consider the system S in Example 1, and two restrictions $S_1 = S_k^n$ and $S_2 = S_l^l$ constructed as in Example 5. The pullback of S_1 and S_2 is $S_{12} = S_k^l$ (defined as in Example 5 if $k \leq l$, or the system with the empty set of control variables and actions if $l < k$). This pull-back describes the interaction between the systems S_1 and S_2 : If $k \leq l$, the local controllers of S_1 and S_2 “synchronize” on this shared subsystem and behave both as a controller for S_{12} . If $k > l$ there is no synchronization between the controllers for S_1 and S_2 ; they can be considered to be independent. Both situations are represented in Fig. 3.

The colimit \overline{S} of the diagram $\{S_1, S_2, S_{12}\}$ (with transition-connected morphisms $S_k^l \hookrightarrow S_k^n, S_k^l \hookrightarrow S_l^l$) has the following components:

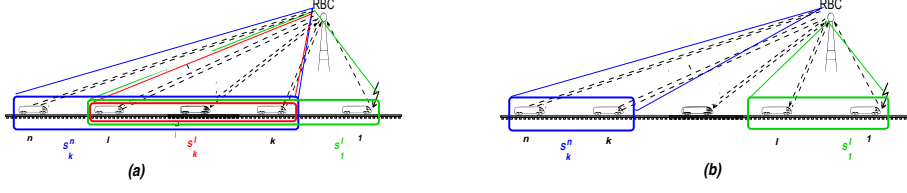


Fig. 3. The systems S_k^n, S_k^l, S_1^l for (a) $k \leq l$ and (b) $l < k$

- $\Sigma_{\bar{S}} = \Sigma; M_{\bar{S}} = M;$
- $X_{\bar{S}} = \bigcup_{i \in \{1, \dots, l\} \cup \{k, \dots, n\}} \{\text{TrainIndex}_i, \text{ActualPos}_i, \text{RepPos}_i, \text{Mode}_i\},$
- $\Gamma_{\bar{S}} = \{\text{succ}(\text{TrainIndex}_i) = \text{TrainIndex}_{i+1} \mid i \in \{1, \dots, l-1\} \cup \{k, \dots, n-1\}\}^\bullet,$
- $A_{\bar{S}} = \bigcup_{i \in \{1, \dots, l\} \cup \{k, \dots, n\}} \{\text{report}_i, \text{move}_i\} \cup \{\text{update}\};$
- $C_{\bar{S}} = (\{\text{report}_i = \text{update} \mid i \in \{1, \dots, l\} \cup \{k, \dots, n\}\} \cup \{\text{report}_i \wedge \text{move}_i = 0 \mid i \in \{1, \dots, l\} \cup \{k, \dots, n\}\} \cup \{\text{move}_1 = \dots = \text{move}_l\} \cup \{\text{move}_k = \dots = \text{move}_n\})^\bullet.$

If $k \leq l$ then \bar{S} coincides with S . If $l < k - 1$ then $X_{\bar{S}} \neq X$, so \bar{S} is obviously different from S . Assume now that $l = k - 1$. Then $X_{\bar{S}} = X, A_{\bar{S}} = A, C_{\bar{S}} = C$, but $\Gamma_{\bar{S}} \neq \Gamma$ (the constraint $\text{succ}(\text{TrainIndex}_{k-1}) = \text{TrainIndex}_k$ cannot be recovered from $\Gamma_1^l \cup \Gamma_k^n$), hence \bar{S} is different from S also in this case.

3 Sheaf theory: definitions

Our goal is to express the links between components of a system and the result of their interconnection. We start from the observation that compatible local states can be 'glued' into a global state (similar for parallel actions, transitions). For expressing such gluing condition in a general setting, we use sheaf theory. In this section we present the main definitions and results in sheaf theory. We start with a presentation of (pre)sheaves and bundles and the relationships between them, and then we continue with some results concerning sheaves of algebras.

In what follows, basic notions from category theory are assumed to be known. For details we refer to [9] or to [12]. Categories and sheaves will be denoted in sans-serif style, e.g. $\text{Set}, \text{Sh}(I)$.

3.1 Presheaves, Sheaves, Bundles

Let I be a topological space, and $\Omega(I)$ the topology on I .

Definition 7 A presheaf on I is a functor $P : \Omega(I)^{op} \rightarrow \text{Sets}$. Let $U \subseteq V$ be open sets in I , and $i_U^V : U \hookrightarrow V$ the inclusion morphism in $\Omega(I)$. The restriction to U , $P(i_U^V) : P(V) \rightarrow P(U)$ is denoted by ρ_U^V .

A sheaf on I is a presheaf $F : \Omega(I)^{op} \rightarrow \text{Sets}$ that satisfies the following condition:

for each open cover $(U_i)_{i \in I}$ of U and family of elements $s_i \in F(U_i)$
s.t. for all i, j , $\rho_{U_i \cap U_j}^{U_i}(s_i) = \rho_{U_i \cap U_j}^{U_j}(s_j)$, there is a unique $s \in F(U)$ with
 $\rho_{U_i}^U(s) = s_i$ for all i .

The morphisms of (pre)sheaves are natural transformations. We denote by $\text{PreSh}(I)$ the category of presheaves over I and by $\text{Sh}(I)$ the category of sheaves over I .

Definition 8 The stalk of a sheaf F on I at a point $i \in I$ is the colimit $F_i = \varinjlim_{U \ni i} F(U)$, where U ranges over all open neighborhoods of i . The assignment $F \mapsto F_i$ defines the stalk functor at i , $\text{Stalk}_i : \text{Sh}(I) \rightarrow \text{Set}$.

Sheaves can be defined also in a different way. An indexed system of sets $(F_i)_{i \in I}$ can alternatively be regarded as a map $f : F = \coprod_{i \in I} F_i \rightarrow I$, with the property that for every $x \in F$, $f(x) = i$ if and only if $x \in F_i$. If the index set I has a topology, then the set F can be endowed with a topology such that f is continuous (i.e. the sets in the family $(F_i)_{i \in I}$ are continuously indexed).

Definition 9 A bundle over I is a triple (F, f, I) where F and I are topological spaces and $f : F \rightarrow I$ is continuous. For every $i \in I$, $f^{-1}(i)$ will be denoted by F_i . Then $F = \coprod_{i \in I} F_i$. Let (F, f, I) and (G, g, I) be two bundles over I . A morphism between (F, f, I) and (G, g, I) is a continuous map $h : F \rightarrow G$ such that $g \circ h = f$.

The category of bundles over I is denoted Sp/I .

Let LH/I be the full subcategory of Sp/I with objects (F, f, I) , where $f : F \rightarrow I$ is a local homeomorphism (i.e. for every $a \in F$ there are open neighborhoods U and U' of a respectively $f(a)$ such that $f : U \rightarrow U'$ is a homeomorphism).

Definition 10 Let (F, f, I) be a bundle over I . A partial section defined on an open subset $U \subseteq I$ is a continuous map $s : U \rightarrow F$ with the property that $f \circ s$ is the inclusion $U \subseteq I$. A section defined on I is called global section. The set of all partial sections over the open subset U of I will be denoted by $\Gamma(F, f)(U)$.

The following links between (pre)sheaves and bundles exist:

- For every bundle (F, f, I) let $\Gamma(F) = \{s : I \rightarrow F \mid s \text{ continuous and } f \circ s = \text{id}_I\}$, the set of all global sections of F . This defines a functor

$$\Gamma : \text{Sp}/I \rightarrow \text{PreSh}(I).$$

- Let F be a presheaf on I . For every $i \in I$ let F_i be the stalk of F at a point $i \in I$. The collection of stalks $(F_i)_{i \in I}$ is an I -indexed family of sets. Let $D(F)$ denote the disjoint union of the stalks, and let $\pi : D(F) \rightarrow I$ be the canonical projection on I defined by $\pi(x) = i$ iff $x \in F_i$. For $s \in F(U)$ and $i \in U$, let s_i be the image of s in F_i . The map $\bar{s} : U \rightarrow D(F)$, $\bar{s}(i) = s_i$ defines a partial section of $\pi : D(F) \rightarrow I$; we impose on $D(F)$ the coarsest topology for which all such sections are continuous. $D(F) = (D(F), \pi, I)$ is a bundle. This construction defines a functor

$$D : \text{PreSh}(I) \rightarrow \text{Sp}/I.$$

Theorem 11 (cf. [9,12]) *The functor $D : \text{PreSh}(I) \rightarrow \text{Sp}/I$ preserves finite limits and is left adjoint to $\Gamma : \text{Sp}/I \rightarrow \text{PreSh}(I)$. The functors D, Γ restrict to an equivalence of categories between $\text{Sh}(I)$ and LH/I .*

$\Gamma \circ D : \text{PreSh}(X) \rightarrow \text{Sh}(X)$ is known as the *sheafification functor*.

Theorem 12 (cf. [9,12]) *The inclusion $\text{Sh}(X) \rightarrow \text{PreSh}(X)$ has a left adjoint, $\Gamma \circ D : \text{PreSh}(X) \rightarrow \text{Sh}(X)$. The sheafification functor $\Gamma \circ D$ preserves all finite limits.*

3.2 Sheaves of algebras

Let A be an algebra of similarity type Σ (no predicate symbols are contained in the signature), $(\theta_i)_{i \in I}$ a family of congruences on A , and τ a topology on I . The following problem was addressed and solved in [2]: In which situation does a sheaf exist with fibers $A_i = A/\theta_i$ such that for every $a \in A$ the map $[a] : I \rightarrow \coprod_{i \in I} A_i$ is a global section? Two constructions are possible:

Construction 1 Let (F_A, f, I) be defined by $F_A = \coprod_{i \in I} A/\theta_i$, and $f : F_A \rightarrow I$ be the natural projection. Assume that a subbasis for the topology on F_A is $\{[a](U) \mid U \in \tau, a \in A\}$, where $[a](U) = \{[a](i) \mid i \in U\} = \{[a]_{\theta_i} \mid i \in U\}$.

Construction 2 Let $G_A : \tau \rightarrow \text{Str}_\Sigma$ be defined on objects by $G_A(U) = A/\theta_U$, where $\theta_U = \bigwedge_{i \in U} \theta_i$ and on morphisms, for every $V \subseteq U$ by the canonical morphism $G_A(U) = A/\theta_U \rightarrow A/\theta_V = G_A(V)$, $a_{\theta_U} \mapsto a_{\theta_V}$.

Let $G_i = \varinjlim_{i \in U} G_A(U)$ be the stalks of G_A , and for every $i \in I$ let $g_i : G_i \rightarrow A_i$ be the unique morphism that arises from the universality property of the colimit. Note that $g_i(\rho_i^U(a)) = a_{\theta_i}$ for every $U \in \tau$ and every $i \in I$. G_A is a presheaf of algebras. Let (SG_A, g, I) be the associated sheaf.

In Construction 1, the stalk at i is isomorphic to A_i , but (F_A, f, I) might be not a sheaf space. In Construction 2, (SG_A, g, I) is a sheaf space, but $g_i : G_i \rightarrow A_i$ may not be an isomorphism.

Theorem 13 ([2]) *The following conditions are equivalent:*

- (1) *If $[a]_{\theta_i} = [b]_{\theta_i}$ then there is an open neighborhood U of i such that for every $j \in U$, $[a]_{\theta_j} = [b]_{\theta_j}$.*
- (2) *(F_A, f, I) is a sheaf of algebras.*
- (3) *For every $i \in I$, $g_i : G_i \rightarrow A_i$ is an isomorphism.*

Definition 14 *If $(\theta_i)_{i \in I}$ is a family of congruences on an algebra A , then any topology on I that satisfies (1) is called an S -topology.*

Corollary 15 ([2]) *Assume that the topology on I is an S -topology with respect to the family of congruences $(\theta_i)_{i \in I}$. Then (F_A, f, I) and (SG_A, g, I) are isomorphic sheaves of algebras for which*

- (1) *The stalk at i is isomorphic to $A_i = A/\theta_i$,*

- (2) The map $\alpha : A \rightarrow \Gamma(I, F_A)$ defined by $\alpha(a) = ([a]_{\theta_i})_{i \in I}$ is a homomorphism,
- (3) In $A \xrightarrow{\alpha} \Gamma(I, F_A) \leq \prod_{i \in I} A/\theta_i \xrightarrow{p_i} A/\theta_i$:
- (i) $p_i \circ \alpha$ is an epimorphism, and
 - (ii) A is a subdirect product of the family $(A/\theta_i)_{i \in I}$ iff $\bigwedge_{i \in I} \theta_i = \Delta_A$ (i.e. iff α is a monomorphism).

The coarsest S-topology on I can be constructed as follows:

Lemma 16 ([2], [10]) *Let $A \hookrightarrow \prod_{i \in I} A_i \xrightarrow{p_i} A_i$ be a subdirect product. The coarsest S-topology on I is generated by the sets $E(a, b) = \{i \in I \mid p_i(a) = p_i(b)\}$ as a subbasis.*

Lemma 17 ([10]) *Let $A \hookrightarrow \prod_{i \in I} A_i \xrightarrow{p_i} A_i$ be a subdirect product and τ_1, τ_2 be two topologies on I . If $\tau_1 \subseteq \tau_2$ and τ_1 contains the equalizer topology induced by A (generated by the sets $E(a, b)$ as a subbasis), then $\Gamma(F_A, (I, \tau_1)) \subseteq \Gamma(F_A, (I, \tau_2))$.*

Even if the topology on I is an S-topology, A is not necessarily isomorphic to the algebra $\Gamma(I, F_A)$. A necessary and sufficient condition for A to be isomorphic to an algebra of global sections of a sheaf with fibers $A_i = A/\theta_i$, for $i \in I$ is given below:

Definition 18 *A family $(c_i)_{i \in I}$ of elements of A is said to be global with respect to $(\theta_i)_{i \in I}$ if for every $i \in I$ there exist $a_1^i, \dots, a_n^i, b_1^i, \dots, b_n^i \in A$ such that:*

- (i) $(a_j^i, b_j^i) \in \theta_i$ for every $j = 1, \dots, n$,
- (ii) If $(a_j^i, b_j^i) \in \theta_k$ for every $j = 1, \dots, n$ then $(c_k, c_i) \in \theta_k$.

Theorem 19 ([2]) *Let $(\theta_i)_{i \in I}$ be a family of congruences on an algebra A such that A is a subdirect product of $(A/\theta_i)_{i \in I}$. Endow I with its coarsest S-topology. Then $\alpha : A \rightarrow \Gamma(I, F_A)$ is an isomorphism iff for every family of elements $(c_i)_{i \in I}$ global with respect to $(\theta_i)_{i \in I}$, there is a $c \in A$ with $(c, c_i) \in \theta_i$ for every $i \in I$.*

4 Interacting Systems

When analyzing concrete complex systems, we tend to be interested in a subcategory of TcSys , containing only the systems relevant for a given application. To this end, we assume a family InSys of interacting systems is specified, fulfilling:

1. All $S \in \text{InSys}$ are t.c. subsystems of a system \overline{S} with $A_{\overline{S}}$ finite.
2. InSys is closed under all pullbacks $S_1 \wedge S_2$ of t.c. subsystems S_1, S_2 of \overline{S} .
3. (InSys, \wedge) is a meet-semilattice.

The first condition enforces the compatibility of models on common sorts and the finiteness of A_S for every $S \in \text{InSys}$; the second and third condition ensure that all systems by which communication is handled are taken into account. A system obtained by interconnecting some elements of InSys can either be seen as the set of all elements of InSys by whose interaction it arises (a subset of InSys which is downwards-closed with respect to \hookrightarrow) or as the colimit of such a family of elements.

We define $\Omega(\text{InSys})$ as consisting of all families of elements of InSys which are closed under transition connected subsystems. Clearly, $\Omega(\text{InSys})$ is a topology on InSys . In fact, $\Omega(\text{InSys})$ is the Alexandroff topology associated with the dual of the poset $(\text{InSys}, \hookrightarrow)$. Since we assumed that InSys is finite and closed under pullbacks, this topology coincides with the Scott topology associated with the dual of $(\text{InSys}, \hookrightarrow)$.

Example 7 Consider now the extension of the example in Section 1 considered in Example 6: Let $k, l \in \{1, \dots, n\}$ with $k \leq l$, and let:

- $I_1 = \{k, \dots, n\}, I_2 = \{1, \dots, l\}, I_{12} = \{k, \dots, l\}$.
- $\text{InSys} = \{S_1, S_2, S_{12}\}$ be the family consisting of the subsystems of $S = (\Sigma, X, \Gamma, M, A, C)$ described in Section 1 corresponding to the sets of trains with indices in I_1, I_2 and I_{12} respectively: $S_1 = S_k^n, S_2 = S_1^l, S_{12} = S_k^l$ (cf. Fig. 4).

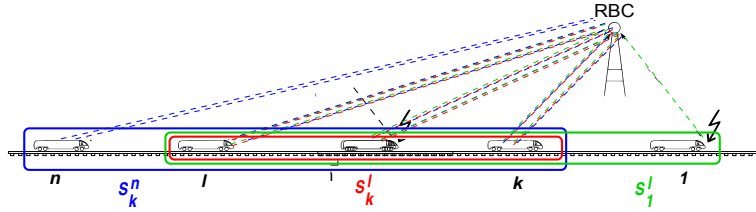


Fig. 4. Case $k \leq l$: S_1^l, S_k^n, S_k^l

Then InSys satisfies conditions (1), (2) and (3) above. The system obtained by interconnecting S_1, S_2, S_{12} can be regarded either as the set $\{S_1, S_2, S_{12}\}$ or as the colimit of the diagram defined by these systems, which coincides with the system S defined in Section 1. In this case, $\Omega(\text{InSys})$ consists of the following sets $\{\emptyset, \{S_{12}\}, \{S_1, S_{12}\}, \{S_2, S_{12}\}, \{S_1, S_2, S_{12}\}\}$.

Our goal is to express the links between components of a system and the result of their interconnection. We start from the observation that compatible local states can be 'glued' into a global state (similar for parallel actions, transitions). For expressing such gluing condition in a general setting, we use sheaf theory.

5 States, partial actions

Let InSys be a family of systems satisfying conditions (1), (2), (3) at the beginning of Section 4, and $\Omega(\text{InSys})$ be the topology on InSys consisting of all subsets InSys which are closed under t.c. subsystems. We define functors modeling states and parallel actions:

(St) $\text{St} : \Omega(\text{InSys})^{\text{op}} \rightarrow \text{Set}$ is defined as follows:

Objects: $\text{St}(U) = \{(s_i)_{S_i \in U} \mid s_i \in \text{St}(S_i), \text{ and if } S_i \hookrightarrow S_j \text{ then } s_i = s_j|_{X_i}\}$;

Morphisms: if $U_1 \stackrel{L}{\subseteq} U_2$, $\text{St}(U_2) \rightarrow \text{St}(U_1)$ is $\text{St}(U_2|_{U_1}) = (s_i)_{S_i \in U_1}$.

(Pa) $\text{Pa} : \Omega(\text{InSys})^{\text{op}} \rightarrow \text{Set}$ is defined as follows:

Objects: $\text{Pa}(U) = \{(f_i)_{S_i \in U} \mid f_i \in \text{Pa}(S_i), \text{ and if } S_i \hookrightarrow S_j \text{ then } f_i = f_j|_{A_i}\}$;

Morphisms: if $U_1 \stackrel{L}{\subseteq} U_2$, $\text{Pa}(U_2) \rightarrow \text{Pa}(U_1)$ is $\text{Pa}(U_2|_{U_1}) = (f_i)_{S_i \in U_1}$.

Example 8 Consider the family $\text{InSys} = \{S_1, S_{12}, S_2\}$ in Example 6. We illustrate the notion of state and partial action defined above.

States. Any tuple (s_1, s_2, s_{12}) , where $s_i \in \text{St}(S_i)$ for $i \in \{1, 2, 12\}$ and $s_1|_{X_{12}} = s_2|_{X_{12}} = s_{12}$, is an element in $\text{St}(\text{InSys})$. Assume first that $k \leq l$.

- Let $s_i : X_{S_i} \rightarrow M$ be such that $s(\text{TrainIndex}_i) = i$ for all $i \in \{1, \dots, l\}$, and such that $s_1|_{X_{12}} = s_2|_{X_{12}} = s_{12}$. Then $(s_1, s_2, s_{12}) \in \text{St}(\text{InSys})$.
- Let $s_1 : X_{S_1} \rightarrow M$ be defined by $s(\text{TrainIndex}_i) = i$ for all $i \in \{1, \dots, l\}$, and $s_2 : X_{S_2} \rightarrow M$ be defined by $s(\text{TrainIndex}_i) = i + 1$ for all $i \in \{k, \dots, n\}$. $s_1 \in \text{St}(S_1)$, $s_2 \in \text{St}(S_2)$, but they do not agree on the common control variables (in particular, $s_1(\text{TrainIndex}_k) = k$, $s_2(\text{TrainIndex}_k) = k + 1$). So $(s_1, s_2, s_1|_{X_{S_{12}}}) \notin \text{St}(\text{InSys})$.

Assume now that $l < k$. Then S_{12} is the system with an empty set of control variables. Hence, $s_1 : X_{S_1} \rightarrow M$ defined by $s(\text{TrainIndex}_i) = i$ for all $i \in \{1, \dots, l\}$, and $s_2 : X_{S_2} \rightarrow M$, defined by $s(\text{TrainIndex}_i) = i + 1$ for all $i \in \{k, \dots, n\}$, agree on the common variables. Therefore $(s_1, s_2, s_1|_{X_{S_{12}}}) \in \text{St}(\text{InSys})$.

Let $U = \{S_1, S_{12}, S_2\}$ and $U_1 = \{S_1, S_{12}\}$ be the two sets in $\Omega(\text{InSys})$ which contain S_1 , and let i be the inclusion between U_1 and U . Then $\text{St}(i) : \text{St}(U) \rightarrow \text{St}(U_1)$ is defined by $\text{St}(i)(s_1, s_2, s_{12}) = \rho_{U_1}^U(s_1, s_2, s_{12}) = (s_1, s_{12})$.

Parallel Actions. Any tuple (f_1, f_2, f_{12}) , where $f_i \in \text{Pa}(S_i)$ for $i \in \{1, 2, 12\}$ and $f_1|_{A_{12}} = f_2|_{A_{12}} = f_{12}$, is an element in $\text{Pa}(\text{InSys})$. In particular:

- (f_1, f_2, f_{12}) with $f_j^{-1}(1) = \{\text{report}_i \mid i \in I_j\} \cup \text{update}$. These are admissible parallel actions in the corresponding systems, and $f_1|_{A_{12}} = f_2|_{A_{12}} = f_{12}$. Then $(f_1, f_2, f_{12}) \in \text{Pa}(\text{InSys})$.

Tuples (f_1, f_2, f_{12}) which do not satisfy these conditions are not in $\text{Pa}(\text{InSys})$:

- (f_1, f_2, f_{12}) with $f_j^{-1}(1) = \{\text{report}_i \mid i \in I_j\} \cup \text{update} \cup \{\text{move}_i \mid i \in I_j\}$ is not in $\text{Pa}(\text{InSys})$, because the components are not admissible parallel actions.

- (f_1, f_2, f_{12}) with $f_1^{-1}(1) = \{\text{report}_i \mid i \in I_1\} \cup \text{update}$ and $f_2^{-1}(1) = \{\text{move}_i \mid i \in I_2\}$ is not in $\text{Pa}(\text{InSys})$, because the components do not agree on A_{12} .

We now show that St and Pa are sheaves and describe the form of their stalks and global sections.

Below, for every $S_i \in \text{InSys}$, the set of all elements in InSys contained in S_i is denoted by $\downarrow S_i$.

Lemma 20 *Let $F : \Omega(\text{InSys})^{op} \rightarrow \text{Set}$ be a sheaf. Then for every $S_i \in \text{InSys}$, the stalk of F at S_i , F_{S_i} , is in bijective correspondence with $F(\downarrow S_i)$.*

Proof: (Sketch) We know that the stalk of F at S_i is $F_{S_i} = \varinjlim_{S_i \in U} F(U)$. Clearly, $\downarrow S_i \subseteq U$ if U is a downwards-closed set with $S_i \in U$. It is easy to show that $F(\downarrow S_i)$ satisfies the universality property of the colimit of the diagram $\{F(U) \mid S_i \in U\}$ (with the appropriate morphisms $F(U) \xrightarrow{\rho_V^U} F(V)$ for every $V \subseteq U$). Therefore, there is a bijection between F_{S_i} and $F(\downarrow S_i)$. \square

Lemma 21 *Let $\mathcal{L}_1 = (\Sigma_1, X_1)$ and $\mathcal{L}_2 = (\Sigma_2, X_2)$ be two languages such that \mathcal{L}_1 is a sublanguage of \mathcal{L}_2 (i.e. $\Sigma_1 \subseteq \Sigma_2$ and $X_1 \subseteq X_2$). Let $M_1 \in \text{Str}_{\Sigma_1}$ and $M_2 \in \text{Str}_{\Sigma_2}$ be such that $M_1 = U_{\Sigma_1}^{\Sigma_2}(M_2)$ (the restriction of M_2 to the signature Σ_1).*

Let ϕ be a formula in the language \mathcal{L}_1 (i.e. containing only operation symbols, predicate symbols and variables from \mathcal{L}_1). Then for every (sort-preserving) assignment $s : X_2 \rightarrow M_2$, s satisfies ϕ if and only if its restriction to X_1 , $s|_{X_1} : X_1 \rightarrow M_1$, satisfies ϕ .

Proof (Sketch): Clearly, if $s : X_2 \rightarrow M_2$ then its restriction to X_1 indeed has values in M_1 , i.e. $s|_{X_1} : X_1 \rightarrow M_1$. By structural induction, $\bar{s}(t) = \bar{s}|_{X_1}(t) \in M_1$ for every term t in the language \mathcal{L}_1 , where \bar{s} is the unique extension of s to terms. Also by structural induction (w.r.t. the structure of formulae), for every assignment $s : X_2 \rightarrow M_2$ and every formula ϕ in the language \mathcal{L}_1 , s satisfies ϕ if and only if its restriction to X_1 satisfies ϕ (for a detailed proof see [17], Sect. 3.3, Lemma 3.19). \square

Lemma 22 *Let $S = (\Sigma, X, \Gamma, M, A, C)$ be the colimit of the diagram defined by the family $\{S_j \mid j \in J\}$ of systems (all being subsystems of a system \bar{S}). We assume that $\{S_j \mid j \in J\}$ is closed under pullbacks, and that for every $j \in J$, $S_j = (\Sigma_j, X_j, \Gamma_j, M_j, A_j, C_j)$.*

- (1) *Let $\{s_j \mid j \in J\}$ be a matching family of elements $s_j \in \text{St}(S_j)$ (i.e. such that for all $j_1, j_2 \in J$, $s_{j_1}|_{X_{j_1} \cap X_{j_2}} = s_{j_2}|_{X_{j_1} \cap X_{j_2}}$). Then there exists a unique $s \in \text{St}(S)$ such that $s|_{X_j} = s_j$ for every $j \in J$.*
- (2) *Let $\{f_i \mid i \in I\}$ be a family of parallel actions $f_i \in \text{Pa}(S_i)$ such that for all $i, j \in I$, $f_i|_{A_i \cap A_j} = f_j|_{A_i \cap A_j}$. Then there exists a unique $f \in \text{Pa}(S)$ such that $f|_{A_i} = f_i$.*

Proof: (1) Define $s : X \rightarrow M$, by $s(x) = s_j(x)$ if $x \in X_j$. Let $\phi \in \bigcup \Gamma_j$. Then ϕ contains exclusively symbols from $\mathcal{L}_j = (\Sigma_j, X_j)$, for some j , hence, by Lemma 21, s satisfies ϕ because s_j , its restriction to X_j , does. Therefore s satisfies all constraints in $\bigcup \Gamma_j$, so also all their consequences: all formulae in $\Gamma = (\bigcup \Gamma_j)^\bullet$. Hence $s \in St(S)$. By the definition of s , s clearly is the unique element of $St(S)$ such that for every $i \in I$, $s|_{X_i} = s_i$. The proof of (2) is similar. \square

Proposition 23 *Let $U = \{S_i \mid i \in I\} \in \Omega(\text{InSys})$, where for every $i \in I$, $S_i = (\Sigma_i, X_i, \Gamma_i, A_i, C_i)$. Let S be the colimit of the diagram defined by U . There are bijective correspondences between $St(S)$ and $\text{St}(U)$, and between $\text{Pa}(S)$ and $\text{Pa}(U)$.*

Proof: By Proposition 6, the colimit of the diagram defined by $U = \{S_i \mid i \in I\}$ is the system S , with $\Sigma_S = \bigcup_{i \in I} \Sigma_i$, $X_S = \bigcup_{i \in I} X_i$, $M_S = M_U|_{\Sigma_S}$, $\Gamma_S = (\bigcup_{i \in I} \Gamma_i)^\bullet$, $A_S = \bigcup_{i \in I} A_i$, $C_S = (\bigcup_{i \in I} C_i)^\bullet$.

Let $\pi : St(S) \rightarrow \text{St}(U)$ be defined for every $s \in St(S)$ by $\pi(s) = (s|_{X_i})_{S_i \in U}$. By Lemma 21, for every $i \in I$, $s_i = s|_{X_i} : X_i \rightarrow M_i \in St(S_i)$. Moreover, if $S_i \hookrightarrow S_j$ then $s_j|_{X_i} = s_i$. So π is well-defined.

Conversely, since InSys (hence, any $U \in \Omega(\text{InSys})$) is closed under pullbacks, by Lemma 22, for every $(s_i)_{S_i \in U}$ with $s_i \in St(S_i)$ and such that if $S_i \hookrightarrow S_j$, $s_i = s_j|_{X_j}$, there exists a unique $s \in St(S)$ with $s|_{X_i} = s_i$, for every $i \in I$. Therefore π is bijective. A similar argument applies for Pa . \square

Corollary 24 *For every $S_i \in \text{InSys}$ there are bijective correspondences between the stalk St_{S_i} at S_i and $St(S_i)$, and between the stalk Pa_{S_i} at S_i and $\text{Pa}(S_i)$.*

Proof: Follows from Lemma 20, Prop. 23, and the fact that the colimit of the diagram defined by $\downarrow S_i$ is S_i . \square

Theorem 25 *The functors St and Pa are sheaves on InSys . For each $S_i \in \text{InSys}$, the stalk at S_i of St (resp. Pa) is in bijection with $St(S_i)$ (resp. $\text{Pa}(S_i)$). Moreover, for each $U \in \Omega(\text{InSys})$, $\text{St}(U)$ (resp. $\text{Pa}(U)$) is in bijection with $St(S_U)$ (resp. $\text{Pa}(S_U)$), where S_U is the colimit of the diagram defined by U .*

Proof: The fact that $\text{St}, \text{Pa} : \Omega(\text{InSys})^{\text{op}} \rightarrow \text{Set}$ are sheaves follows immediately from the definition of a sheaf. The fact that for every $S_i \in \text{InSys}$, the stalk of St at S_i , St_{S_i} is in bijective correspondence with $\text{St}(\downarrow S_i)$ and with $St(S_i)$; and the stalk of Pa at S_i , Pa_{S_i} is in bijective correspondence with $\text{Pa}(\downarrow S_i)$ and with $\text{Pa}(S_i)$ follows from Lemma 20 and Corollary 24, taking into account the fact that St and Pa are sheaves. The fact that for each $U \in \Omega(\text{InSys})$, $\text{St}(U)$ (resp. $\text{Pa}(U)$) is in bijection with $St(S_U)$ (resp. $\text{Pa}(S_U)$), where S_U is the colimit of the diagram defined by U follows from Proposition 23. \square

Example 9 Let $\text{InSys} = \{S_1, S_{12}, S_2\}$ as defined in Example 7 (with $k \leq l$):

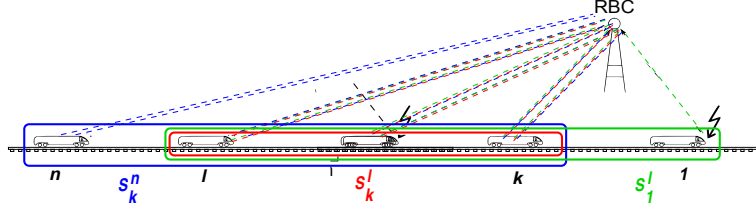


Fig. 5. Case $k \leq l$: $S_1 = S_1^l, S_2 = S_k^n, S_{12} = S_k^l$

- (1) An example of an open cover for $U = \{S_1, S_2, S_{12}\}$ is $\{U_1, U_2, U_{12}\}$, where $U_1 = \{S_1, S_{12}\}, U_2 = \{S_2, S_{12}\}, U_{12} = \{S_{12}\}$. Let $(s_1, s_{12}) \in St(U_1)$ and $(t_2, t_{12}) \in St(U_2)$ be such that $\rho_{U_{12}}^{U_1}(s_1, s_{12}) = \rho_{U_{12}}^{U_2}(t_2, t_{12})$. Then $s_{12} = t_{12}$ and there is a unique element $(s_1, t_2, s_{12}) \in St(U)$ such that $\rho_{U_1}^U(s_1, t_2, s_{12}) = (s_1, s_{12})$ and $\rho_{U_2}^U(s_1, t_2, s_{12}) = (t_2, t_{12})$. Similar for Pa .
- (2) The stalk of St at S_1 is the colimit of the diagram

$$St(U) \xrightarrow{St(i)} St(U_1) \xrightarrow{St(id)} St(U_1)$$

and hence in bijection with $St(U_1)$. Similarly for Pa .

- (3) It can be seen that $St(U)$ is in bijection with $St(S)$, where S is the system in the example in Section 1: Let $(s_1, s_2, s_{12}) \in St(U)$. Then $s : X \rightarrow M$ defined by $s(x) = s_i(x)$ iff $x \in X_i$ is well defined (due to the definition of $St(U)$) and in $St(S)$. Conversely, if $s \in St(S)$, then $(s_{X_1}, s_{X_2}, s_{X_{12}}) \in St(U)$. Also $Pa(U)$ is in bijection with $Pa(S)$: If $(f_1, f_2, f_{12}) \in Pa(U)$ then $f : A \rightarrow \{0, 1\}$ defined by $f(x) = f_i(x)$ iff $x \in A_i$ is well defined (due to the definition of $Pa(U)$). It can also be checked that if $f_1 \models C_1$ and $f_2 \models C_2$ then $f \models C$. Thus, $f \in Pa(S)$. The converse is immediate.

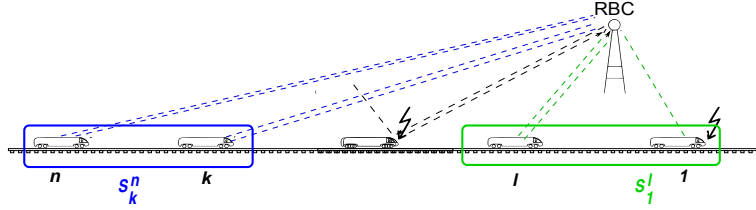


Fig. 6. Case $k > l$: $S_1 = S_1^l, S_2 = S_k^n, S_{12} = S_k^l$

Assume now that S_1, S_2, S_{12} are as in Example 6 but $l < k$, say $l = k - 1$. The open cover and stalk construction in (1) and (2) above are the same. However, $St(U)$ is in bijection with $St(\bar{S})$, where \bar{S} is the colimit of the diagram defined by U as described in Example 6 which in this case is different from S . In particular, $s : X \rightarrow M$ with $s(\text{TrainIndex}_1) = 1, s(\text{TrainIndex}_2) = 2, \dots, s(\text{TrainIndex}_{k-1}) =$

$k - 1$ and $s(\text{TrainIndex}_k) = k + 1, \dots, s(\text{TrainIndex}_{n-1}) = n$ is a state of \overline{S} , but not of S .

6 Transitions

Let InSys be a family of systems satisfying conditions (i), (ii), (iii) in Section 4. We define a functor modeling transitions:

(Tr) $\text{Tr} : \Omega(\text{InSys})^{\text{op}} \rightarrow \text{Set}$ is defined as follows:

Objects: $\text{Tr}(U) = \{(f, s, s') \mid f = (f_i)_{S_i \in U} \in \text{Pa}(U), s = (s_i)_{S_i \in U} \in \text{St}(U), s' = (s'_i)_{S_i \in U} \in \text{St}(U), (s_i, s'_i) \in \text{Tr}_{S_i}(f_i), \text{ for all } S_i \in U\}$;

Morphisms: if $U_1 \stackrel{\iota}{\subseteq} U_2$, $\text{Tr}(\iota) : \text{Tr}(U_2) \rightarrow \text{Tr}(U_1)$ is defined by $\text{Tr}(\iota)((f, s, s')) = (\text{Pa}(\iota)(f), \text{St}(\iota)(s), \text{St}(\iota)(s'))$,

where, for every S_i in InSys and $f_i \in \text{Pa}(S_i)$, $\text{Tr}_{S_i}(f_i)$ is the transition relation associated to f_i in S_i as explained in Section 3.

Example 10 Consider the family $\{S_1, S_{12}, S_2\}$ in Example 7. With the notation introduced in Example 7, let:

- $s_j(\text{ActualPos}_i) = a_i$, $s_j(\text{RepPos}_i) = r_i$, $s_j(\text{Mode}_i) = m_i$, for $i \in I_j$;
- f_j be such that $f_j^{-1}(1) = \{\text{report}_i \mid i \in I_j\} \cup \text{update}$, and
- s'_j be defined by: $s'_j(\text{ActualPos}_i) = a_i$, $s'_j(\text{RepPos}_i) = a_i$, $s'_j(\text{Mode}_i) = m'_i$, where m'_i is computed according to the transition rules for update in Example 1.

Then: $f_i \in \text{Pa}(S_i)$, $s_i, s'_i \in \text{St}(S_i)$, $(s_i, s'_i) \in \text{Tr}(S_i)$ for $i \in \{1, 2, 12\}$,
 $f_{1|A_{12}} = f_{2|A_{12}} = f_{12}$ and $s_{1|X_{12}} = s_{2|X_{12}} = s_{12}$.

Hence, $((f_1, s_1, s'_1), (f_2, s_2, s'_2), (f_{12}, s_{12}, s'_{12}))$ is in $\text{Tr}(\text{InSys})$.

Lemma 26 Let $S \hookrightarrow T$ be a t.c. subsystem. The following hold:

- (1) for every $a \in A_S$ and all $s_1, s_2 \in \text{St}(T)$, if $(s_{1|X_T^a}, s_{2|X_T^a}) \in \text{Tr}_T^a$ then $(s_{1|X_S^a}, s_{2|X_S^a}) \in \text{Tr}_S^a$.
- (2) for every $a \in A_S$ and all $s_1, s_2 \in \text{St}(T)$, if $(s_1, s_2) \in \text{Tr}_T(a)$ then $(s_{1|X_S}, s_{2|X_S}) \in \text{Tr}_S(a)$.
- (3) if the transitions associated to admissible parallel actions satisfy either **(Disj)** or **(Indep)**, then for every $f \in \text{Pa}(T)$ and all $s_1, s_2 \in \text{St}(T)$, if $(s_1, s_2) \in \text{Tr}_T(f)$ then $(s_{1|X_S}, s_{2|X_S}) \in \text{Tr}_S(f|_{A_S})$.

Proof: (1) follows by the definition of a t.c. subsystem; (2) follows immediately from (1) and the definitions of $\text{Tr}_T(a)$ and $\text{Tr}_S(a)$.

(3) First assume that the transitions associated to admissible parallel actions satisfy **(Disj)**. Let $f \in \text{Pa}(T)$ and $s_1, s_2 \in \text{St}(T)$ such that $(s_1, s_2) \in \text{Tr}_T(f)$. Then for every $a \in A_T$ with $f(a) = 1$, $(s_{1|X_T^a}, s_{2|X_T^a}) \in \text{Tr}_T^a$, and for every $x \notin \bigcup_{a \in A_T, f(a)=1} X_T^a$, $s_1(x) = s_2(x)$. Let $a \in A_S$ be such that $f(a) = 1$. Then, by (1), $(s_{1|X_S^a}, s_{2|X_S^a}) \in \text{Tr}_S^a$. Let $x \in X_S$ such that $x \notin \bigcup_{a \in A_S, f(a)=1} X_S^a$. Then $x \notin$

$\bigcup_{a \in A_T, f(a)=1} X_T^a$ (otherwise $x \in X_S \cap \bigcup_{a \in A_T, f(a)=1} X_T^a = \bigcup_{a \in A_S, f(a)=1} X_S^a$), hence $s_1(x) = s_2(x)$. This shows that $(s_1|_{X_S}, s_2|_{X_S}) \in Tr_S(f|_{A_S})$.

Now assume that the transitions associated to admissible parallel actions satisfy **(Indep)**. Let $f \in Pa(T)$ and $s_1, s_2 \in St(T)$ such that $(s_1, s_2) \in Tr_T(f)$. Let $\{a_1, \dots, a_n\} = f^{-1}(1)$ (where all a, b such that $a = b \in C_T$ have been identified). Then there are $s_1 = t_0, t_1, \dots, t_{n-1}, t_n = s_2 \in St(T)$ such that for all $i = 1, \dots, n$, $(t_{i-1}, t_i) \in Tr_T(a_i)$. Let now a_i be such that $f(a_i) = 1$. If $a_i \in A_S$ then, by (2), $(t_{i-1}|_{X_S}, t_i|_{X_S}) \in Tr_S(a_i)$. If $a_i \notin A_S$ then, since S is a t.c. subsystem of T , it follows that $X_T^{a_i} \cap X_S = \emptyset$, hence $t_{i-1}|_{X_S} = t_i|_{X_S}$. Therefore, there exist $s_1|_{X_S} = t'_0, t'_1, \dots, t'_{m-1}, t'_m = s_2|_{X_S} \in St(S)$ such that if $\{a'_1, \dots, a'_m\} = \{a_1, \dots, a_n\} \cap A_S$, then for every $i = 1, \dots, m$, $(t'_{i-1}, t'_i) \in Tr_S(a'_i)$; i.e. $(s_1|_{X_S}, s_2|_{X_S}) \in Tr_S(f|_{A_S})$. \square

Lemma 27 *Let $U = \{S_i \mid i \in I\} \in \Omega(\text{InSys})$, where for every $i \in I$, $S_i = (\Sigma_i, X_i, \Gamma_i, A_i, C_i)$. Let S be the colimit of the diagram defined by U . Let $a \in A_S$, and $s_1, s_2 \in St(S)$ be such that if $a \in A_i$ then $(s_1|_{X_i}, s_2|_{X_i}) \in Tr_{S_i}(a)$; otherwise $s_1|_{X_i} = s_2|_{X_i}$. Then $(s_1, s_2) \in Tr_S(a)$.*

Proof: Let $s_1, s_2 \in St(S)$ be such that $(s_1|_{X_i}, s_2|_{X_i}) \in Tr_{S_i}(a)$ for every $i \in I$. Then, for every $S_i \in U$, if $a \in A_i$ then $(s_1|_{X_i \cap X_S^a}, s_2|_{X_i \cap X_S^a}) \in Tr_{S_i}^a$ and $s_1(x) = s_2(x)$ for all $x \in X_i \setminus X_S^a$, and if $a \notin A_i$ then $s_1|_{X_i} = s_2|_{X_i}$. By the definition of the colimit S it follows that $(s_1|_{X_S^a}, s_2|_{X_S^a}) \in Tr_S^a$. If $x \in X_S \setminus X_S^a$ then $x \notin X_{S_i}^a$ for all $S_i \in U$, hence $s_1(x) = s_2(x)$. \square

Prop. 28 shows that a similar gluing property also holds for parallel actions, if the transitions of composed actions satisfy **(Disj)**.

Proposition 28 *Assume that the transitions associated to admissible parallel actions satisfy **(Disj)**. Let $U = \{S_i \mid i \in I\} \in \Omega(\text{InSys})$, and let S be the colimit of U . Let $s_1, s_2 \in St(S)$ and $f \in Pa(S)$. Assume that $(s_1|_{X_i}, s_2|_{X_i}) \in Tr_{S_i}(f|_{A_i})$ for every $i \in I$. Then $(s_1, s_2) \in Tr_S(f)$.*

Proof: Let $s_1, s_2 \in St(S)$ and $f \in Pa(S)$, and assume that $(s_1|_{X_i}, s_2|_{X_i}) \in Tr_{S_i}(f|_{A_i})$ for every $i \in I$. Let $a \in A_S$ with $f(a) = 1$. Then, by the definition of $Tr_{S_i}(f|_{A_i})$, $(s_1|_{X_S^a \cap X_i}, s_2|_{X_S^a \cap X_i}) \in Tr_{S_i}^a$ for all $i \in I$ with $a \in A_i$, and, for every $x \notin \bigcup_{a \in A_i, f(a)=1} X_i^a$, $s_1(x) = s_2(x)$. Hence, by the definition of the colimit S , $(s_1|_{X_S^a}, s_2|_{X_S^a}) \in Tr_S^a$. For $x \notin \bigcup_{a, f(a)=1} X_S^a$ we know that $s_1|_{X_i}(x) = s_2|_{X_i}(x)$ if $x \in X_i$, hence $s_1(x) = s_2(x)$. Thus, by **(Disj)**, $(s_1, s_2) \in Tr_S(f)$. \square

If we assume that actions may consume common resources and are deterministic, and the transitions of parallel actions are obtained according to **(Indep)**, a similar result holds, as shown in Lemma 29 and Prop. 30.

Lemma 29 *Let $U = \{S_i \mid i \in I\} \in \Omega(\text{InSys})$. Let S be the colimit of U . Assume that the actions in S and $\{S_i \mid i \in I\}$ are deterministic. Let $a \in A_S$ and $s \in St(S)$ be such that for every $i \in I$ there exists a state $s_i \in St(S_i)$ with $(s|_{X_i}, s_i) \in Tr_{S_i}(a)$ (i.e. if $a \notin A_i$ then $s|_{X_i} = s_i$, otherwise $(s|_{X_i \cap X_S^a}, s_i|_{X_i \cap X_S^a}) \in Tr_{S_i}^a$ and $s(x) = s_i(x)$ if $x \in X_i \setminus X_S^a$). Then there is a unique state $\bar{s} \in St(S)$ such that for all $i \in I$, $\bar{s}|_{X_i} = s_i$ and $(s, \bar{s}) \in Tr_S(a)$.*

Proof: Because the actions in S and in $\{S_i \mid i \in I\}$ are deterministic and $\{S_i \mid i \in I\}$ is closed under pullbacks, it follows that for all $i, j \in I$, $s_{i|X_i \cap X_j} = s_{j|X_i \cap X_j}$. Hence, by Lemma 22(1), there exists a unique $\bar{s} \in St(S)$ such that $\bar{s}|_{X_i} = s_i$. It follows that $(s|_{X_i}, \bar{s}|_{X_i}) \in Tr_{S_i}(a)$ for every $i \in I$, with $a \in A_i$ and $s|_{X_i} = \bar{s}|_{X_i}$ otherwise, hence, by Lemma 27, $(s, \bar{s}) \in Tr(a)$. \square

Remark: In proving Lemma 29 we used the fact that U is closed under pullbacks, i.e. U has the property that for all $S_1, S_2 \in U$, the pullback $S_1 \times_{\bar{S}} S_2$ is a t.c. subsystem of both S_1 and S_2 , hence is in U .

The next proposition shows that a gluing condition similar to that of Prop. 28 holds under the assumption that the transitions of parallel actions satisfy (**Indep**). Since in this case one of the conditions is that $f^{-1}(1)$ is finite for all $f \in Pa(S)$, we assume, for the sake of simplicity, that the set of atomic actions of every system S is *finite*. Since we decided to consider only such systems (e.g. by imposing that $A_{\bar{S}}$ is finite, where \bar{S} is as in the description of **lnSys**), this is not a limitation.

Proposition 30 *Let $U = \{S_i \mid i \in I\} \in \Omega(\mathbf{lnSys})$, where for every $i \in I$, $S_i = (\Sigma_i, X_i, \Gamma_i, A_i, C_i)$. Let S be the colimit of the diagram defined by U . Assume that the transitions associated to admissible parallel actions obey (**Indep**). Let $s_1, s_2 \in St(S)$ and $f \in Pa(S)$. Assume $(s_1|_{X_i}, s_2|_{X_i}) \in Tr_{S_i}(f|_{A_i})$ for every $i \in I$. Then $(s_1, s_2) \in Tr_S(f)$.*

Proof: Assume that we have identified all elements $a_1, a_2 \in A_S$ with $a_1 = a_2 \in C_S$ and $f(a_1) = f(a_2) = 1$, and, after this identification, $f^{-1}(1) = \{a_1, \dots, a_n\}$. We proceed by induction on the number n of elements in $f^{-1}(1)$.

If $n = 0$ then $s_1|_{X_i} = s_2|_{X_i}$ for every $i \in I$, hence $s_1 = s_2$. If $n = 1$ then the property is true by Lemma 27. Assume that the property is true for every $g \in Pa(S)$ such that (after identifying all elements $a_1, a_2 \in A_S$ with $a_1 = a_2 \in C_S$ and $g(a_1) = g(a_2) = 1$), the set $g^{-1}(1)$ has $n - 1$ elements. Let $f \in Pa(S)$ with $f^{-1}(1) = \{a_1, \dots, a_n\}$. By (**Indep**)(i) it follows that the parallel action $g : A \rightarrow \{0, 1\}$, with $g(a) = 1$ if and only if $a \in \{a_2, \dots, a_n\}$, is admissible, i.e. $g \in Pa(S)$ and $g|_{A_i} \in Pa(S_i)$, for all $i \in I$. Since $(s_1|_{X_i}, s_2|_{X_i}) \in Tr_{S_i}(f|_{A_i})$ for all $i \in I$, by (**Indep**)(ii) it follows that $f|_{A_i}$ can be applied at $s_1|_{X_i}$ in S_i , for all $i \in I$, and the final state does not depend on the order in which the actions are applied. Therefore we may assume that a_1 is applied first in all systems S_i with $a_1 \in A_i$. Hence, for every $i \in I$ there exists a state $s_i \in St(S_i)$ such that $(s_1|_{X_i}, s_i) \in Tr_{S_i}(a_1)$ if $a_1 \in A_i$, or such that $s_1|_{X_i} = s_i$ if $a_1 \notin A_i$. It also follows that $g|_{A_i}$ can be applied at s_i for every $i \in I$. The final state does not depend on the order in which the actions are applied, hence $(s_i, s_2|_{X_i}) \in Tr_{S_i}(g|_{A_i})$ for all $i \in I$. By Lemma 29 (taking into account the determinism of parallel actions guaranteed by (**Indep**)(ii)) it follows that there is a unique state $\bar{s} \in St(S)$ such that $\bar{s}|_{X_i} = s_i$ for every $i \in I$, and $(s_1, \bar{s}) \in Tr_S(a)$. Moreover, for every $i \in I$, $(\bar{s}|_{X_i}, s_2|_{X_i}) \in Tr_{S_i}(g|_{A_i})$. Therefore, by the induction hypothesis, $(\bar{s}, s_2) \in Tr_S(g)$. This proves that $(s_1, s_2) \in Tr_S(f)$. \square

Proposition 31 $\text{Tr} : \Omega(\text{InSys})^{op} \rightarrow \text{Set}$ defined by

$$\text{Tr}(U) = \{(f, s, s') \mid f = (f_i)_{S_i \in U} \in \text{Pa}(U), s = (s_i)_{S_i \in U}, s' = (s'_i)_{S_i \in U} \in \text{St}(U), \\ (s_i, s'_i) \in \text{Tr}_{S_i}(f_i), \text{ for all } S_i \in U\}$$

is a subsheaf of $\text{Pa} \times \text{St} \times \text{St}$.

Proof: Let $U \in \Omega(\text{InSys})$ and let $(U_j)_{j \in J}$ be an open covering of U .

Let $\{(f^j, s^j, s'^j)\}_{j \in J}$ be such that for every $j \in J$, $(f^j, s^j, s'^j) \in \text{Tr}(U_j)$ and for every j_1, j_2 , $f_i^{j_1} = f_i^{j_2}$, $s_i^{j_1} = s_i^{j_2}$, and $s'^i_{j_1} = s'^i_{j_2}$ for every $S_i \in U_{j_1} \cap U_{j_2}$. From the compatibility conditions in the definition of $\text{Tr}(U_j)$, $j \in J$ it immediately follows that there exists a unique $(f, s, s') \in \text{Tr}(U)$ such that $f_i = f_i^j$ for every $j \in J$ and every $S_i \in U_j$. \square

Proposition 32 Let $U = \{S_i \mid i \in I\} \in \Omega(\text{InSys})$, where for every $i \in I$, $S_i = (\Sigma_i, X_i, \Gamma_i, A_i, C_i)$, and let S be the colimit of the diagram defined by U . Assume that the transitions of parallel actions satisfy either **(Disj)** or **(Indep)**. Then there is a bijective correspondence between $\text{Tr}(S) = \{(f, s, s') \mid (s, s') \in \text{Tr}_S(f)\}$ and $\text{Tr}(U)$.

Proof: For every $(f, s, s') \in \text{Tr}(S)$, $((f|_{A_i})_{S_i \in U}, (s|_{X_i})_{S_i \in U}, (s'|_{X_i})_{S_i \in U}) \in \text{Tr}(U)$. Conversely, let $((f_i)_{S_i \in U}, (s_i)_{S_i \in U}, (s'_i)_{S_i \in U}) \in \text{Tr}(U)$. We know that $(f_i)_{S_i \in U} \in \text{Pa}(U)$ and $(s_i, s'_i) \in \text{Tr}_{S_i}(f_i)$, for all $S_i \in U$. Let $f : A_S \rightarrow \{0, 1\}$ be the unique map that extends f_i , for every $S_i \in U$. Let $s : X_S \rightarrow M_S$ (resp. $s' : X_S \rightarrow M_S$) be the unique map extending s_i (resp. s'_i) for $S_i \in U$, as constructed in Lemma 22. Then, for every $S_i \in U$, $f_i = f|_{A_i}$, $s_i = s|_{X_i}$, $s'_i = s'|_{X_i}$, hence $(s_i, s'_i) = (s|_{X_i}, s'|_{X_i}) \in \text{Tr}_{S_i}(f|_{A_i})$. From Prop. 28 (if the transitions satisfy **(Disj)**), resp. from Prop. 30 (if the transitions satisfy **(Indep)**) it follows that $(s, s') \in \text{Tr}_S(f)$. It is easy to see that the maps between $\text{Tr}(S)$ and $\text{Tr}(U)$ described above are mutually inverse, which proves the claim in the proposition. \square

Corollary 33 For every $S_i \in \text{InSys}$ there is a bijective correspondence between the stalk Tr_{S_i} at S_i and $\text{Tr}(S_i)$.

Proof: Since Tr is a sheaf we know by Lemma 20 that there is a bijective correspondence between the stalk Tr_{S_i} and $\text{Tr}(\downarrow S_i)$. The conclusion follows now from Prop. 32 and the fact that the colimit of the diagram defined by $\downarrow S_i$ is S_i . \square

We summarize the results obtained so far in the following theorem:

Theorem 34 The functor $\text{Tr} : \Omega(\text{InSys})^{op} \rightarrow \text{Set}$ is a subsheaf of $\text{Pa} \times \text{St} \times \text{St}$. Moreover:

- For every $S_i \in \text{InSys}$, the stalk of Tr at S_i is in bijection with $\text{Tr}(S_i) = \{(f, s, s') \mid (s, s') \in \text{Tr}_{S_i}(f)\}$.

- If the transitions obey either **(Disj)** or **(Indep)**, then, for every $U \in \Omega(\text{InSys})$, $\text{Tr}(U)$ is in bijection with $\text{Tr}(S_U) = \{(f, s, s') \mid (s, s') \in \text{Tr}_{S_U}(f)\}$, where S_U is the colimit of the diagram defined by U .

Proof: Follows from Prop. 31, Cor. 33, and Prop. 32. \square

Example 11 Consider the family $\{S_1, S_{12}, S_2\}$ in Example 7. Consider $((f_1, s_1, s'_1), (f_2, s_2, s'_2), (f_{12}, s_{12}, s'_{12})) \in \text{Tr}(U)$. Let $f : A \rightarrow \{0, 1\}$ be defined by $f(x) = f_i(x)$ iff $x \in A_i$ is well defined. Then $f \in \text{Pa}(S)$. Similarly, $s, s' : X \rightarrow M$, defined by $(s(x) = s_i(x) \text{ and } s'(x) = s'_i(x))$ iff $x \in X_i$ are well defined and in $\text{St}(S)$.

As shown in Example 3, the transitions in all systems S_1, S_2, S_{12} obey condition **(Disj)**. The changes of the components of parallel actions are not contradictory and affect only the variables the actions depend upon. Thus, (s, s') is in the transition induced (cf. rule **(Disj)**) by f . Hence, $(s, s') \in \text{Tr}_S(f)$. The converse is an immediate consequence of the fact that, as showed in Example 5, S_1, S_2, S_{12} are transition-connected subsystems of S .

7 Behavior in time

In [6], the behavior of a given system S in time is modeled by a functor $F : \mathcal{T}^{\text{op}} \rightarrow \text{Set}$, where \mathcal{T} is the basis for the topology on \mathbb{N} consisting of all the sets $\{0, 1, \dots, n\}, n \in \mathbb{N}$. Intuitively, for every $T \in \mathcal{T}$, $F(T)$ represents the succession of the states of the systems “observed” during the interval of time T . We analyze various alternative possibilities of modeling behavior.

7.1 Behavior as successions of states and actions.

Since we are interested in actions as well as states, we present a different description of behavior. Let \mathcal{T} consist of \mathbb{N} together with all sets $\{0, 1, \dots, n\}, n \in \mathbb{N}$. The behavior in an interval $T \in \mathcal{T}$ of a complex system obtained by interconnecting a family InSys (satisfying conditions (i)–(iii) in Section 4) is modeled by all successions of pairs (state, action) of the component subsystems that can be observed during T , i.e. by the functor $\mathbf{B}_T : \Omega(\text{InSys})^{\text{op}} \rightarrow \text{Set}$ defined as follows:

$$\begin{aligned} \text{Objects:} \quad & \text{for } U \in \Omega(\text{InSys}), \mathbf{B}_T(U) = \{h : T \rightarrow \text{St}(U) \times \text{Pa}(U) \mid K(h, T)\}, \\ \text{Morphisms:} \quad & \text{for } U_1 \overset{\iota}{\subseteq} U_2 \text{ by } \mathbf{B}_T(\iota) : \mathbf{B}_T(U_2) \rightarrow \mathbf{B}_T(U_1), \text{ where if } h \in \mathbf{B}_T(U_2), \\ & \mathbf{B}_T(\iota)(h) = (\text{St}(\iota) \times \text{Pa}(\iota)) \circ h : T \xrightarrow{h} \text{St}(U_2) \times \text{Pa}(U_2) \xrightarrow{\text{St}(\iota) \times \text{Pa}(\iota)} \text{St}(U_1) \times \text{Pa}(U_1). \end{aligned}$$

Here $K(h, T)$ expresses the fact that for every n , if $n, n+1 \in T$ and $h(n) = (s, f)$, $h(n+1) = (s', f')$ then $(f, s, s') \in \text{Tr}(U)$.

Example 12 We illustrate the definition above. Let $T = \mathbb{N}$, and let $U = \{S_1, S_2, S_{12}\}$ as in Example 7. We represent an element h in $\mathbf{B}_T(\text{InSys})$ as a table (first row: arguments i of h , second row: the value $h(i)$, i.e. a pair of tuples):

| i | $h(i)$ | | | | | | | | | | | | | | | | | |
|-----|----------------------------------|----------------------------------|--------------------------------|---------------------|----------------------------------|----------------------------------|--------------------------------|-------------------------------|-------------------------------|--------------------------------|----------|-------------------------------|-------------------------------|--------------------------------|------------------|-----|-----|-----|
| | $\text{St}(U)$ | | | | | | | | | $\text{Pa}(U)$ | | | | | | | | |
| | $\text{St}(S_1)$ | | | $\text{St}(S_{12})$ | | | $\text{St}(S_2)$ | | | $\text{Pa}(S_1)$ | | | $\text{Pa}(S_{12})$ | | $\text{Pa}(S_2)$ | | | |
| | $(i \in I_1)$ | | | $(i \in I_{12})$ | | | $(i \in I_2)$ | | | $(i \in I_1)$ | | | $(i \in I_{12})$ | | $(i \in I_2)$ | | | |
| | ActPos _{i} | RepPos _{i} | Mode _{i} | (restr.) | ActPos _{i} | RepPos _{i} | Mode _{i} | rep _{i} | upd _{i} | move _{i} | (restr.) | rep _{i} | upd _{i} | move _{i} | | | | |
| 0 | a_i | r_i | m_i | a_i | r_i | m_i | a_i | r_i | m_i | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 0 |
| 1 | a_i | a_i | m_i' | a_i | a_i | m_i' | a_i | a_i | m_i' | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 1 |
| 2 | a_i' | a_i | m_i' | a_i' | a_i | m_i' | a_i' | a_i | m_i' | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 0 |
| 3 | a_i' | a_i' | m_i'' | a_i' | a_i' | m_i'' | a_i' | a_i' | m_i'' | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 0 |
| ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... |

Proposition 35 $B_T : \Omega(\text{InSys})^{op} \rightarrow \text{Set}$ is a sheaf.

Proof: Let $U \in \Omega(\text{InSys})$ and let $(U_j)_{j \in J}$ be a family of elements in $\Omega(\text{InSys})$ be such that $U = \bigcup_{j \in J} U_j$. Let $h_j \in B_T(U_j)$, $j \in J$, be such that for all $i, j \in J$, $\rho_{U_i \cap U_j}^{U_i}(h_i) = \rho_{U_i \cap U_j}^{U_j}(h_j)$. We show that there is a unique $h \in B_T(U)$ such that $\rho_{U_j}^U(h) = h_j$ for every $j \in J$.

From the definition of the behavior of a system, we know that for all $j \in J$ and every $t \in T$, $h_j(t) = (s_j^t, f_j^t)$, where $s_j^t \in \text{St}(U_j)$ and $f_j^t \in \text{Pa}(U_j)$, and if $t, t+1 \in T$ then $(f_j^t, s_j^t, s_j^{t+1}) \in \text{Tr}(U_j)$. We know that for all i, j , $\rho_{U_i \cap U_j}^{U_i}(h_i) = \rho_{U_i \cap U_j}^{U_j}(h_j)$. Therefore, for every $t \in T$, $\rho_{U_i \cap U_j}^{U_i}(h_i)(t) = \rho_{U_i \cap U_j}^{U_j}(h_j)(t)$. This shows that for every $t \in T$ the families $\{s_j^t \in \text{St}(U_j) \mid j \in J\}$ and $\{f_j^t \in \text{Pa}(U_j) \mid j \in J\}$ are such that for all $i, j \in J$, $\rho_{U_i \cap U_j}^{U_i}(s_i^t) = \rho_{U_i \cap U_j}^{U_j}(s_j^t)$ and $\rho_{U_i \cap U_j}^{U_i}(f_i^t) = \rho_{U_i \cap U_j}^{U_j}(f_j^t)$. Since St and Pa are sheaves, it follows that for every $t \in T$ there is a unique $s^t \in \text{St}(U)$ such that $\rho_{U_j}^U(s^t) = s_j^t$ for every $j \in J$, and a unique $f^t \in \text{Pa}(U)$ such that $\rho_{U_j}^U(f^t) = f_j^t$ for every $j \in J$.

Let $h : T \rightarrow \text{St}(U) \times \text{Pa}(U)$ be defined by $h(t) = (s^t, f^t)$ for every $t \in T$. Note that if $t, t+1 \in T$, then $(f_j^t, s_j^t, s_j^{t+1}) \in \text{Tr}(U_j)$ for every $j \in J$. By the proof of Prop. 31, $(f^t, s^t, s^{t+1}) \in \text{Tr}(U)$. Thus, $K(h, T)$ holds, i.e. $h \in B_T(U)$. The uniqueness of h follows easily. \square

Proposition 36 Let $U = \{S_i \mid i \in I\} \in \Omega(\text{InSys})$. If the transitions of parallel actions satisfy either **(Disj)** or **(Indep)** then $B_T(U)$ is in bijection with $B_T(S)$, where S is the colimit of the systems in U .

Proof: We first show that there exists a map $\pi : B_T(U) \rightarrow B_T(S)$.

Let $h \in B_T(U)$, i.e. $h : T \rightarrow \text{St}(U) \times \text{Pa}(U)$ such that $K(h, T)$ holds. Assume that for every $t \in T$, $h(t) = ((s_i^t)_{S_i \in U}, (f_i^t)_{S_i \in U})$. Let $s^t \in \text{St}(S)$ and $f^t \in \text{Pa}(S)$ be obtained analogously to the proof of Prop. 35 (i.e. such that for every $S_i \in U$, $s^t|_{X_i} = s_i^t$ and $f^t|_{A_i} = f_i^t$). Let $\pi(h) : T \rightarrow \text{St}(S) \times \text{Pa}(S)$ be defined by $\pi(h)(t) = (s^t, f^t)$, for every $t \in T$. We show that $\pi(h) \in B_T(S)$.

Let n be such that $n, n+1 \in T$, so $\pi(h)(n) = (s^n, f^n)$ and $\pi(h)(n+1) = (s^{n+1}, f^{n+1})$. We know that $h(n) = ((s^n|_{X_i})_{S_i \in U}, (f^n|_{A_i})_{S_i \in U})$ and $h(n+1) = ((s^{n+1}|_{X_i})_{S_i \in U}, (f^{n+1}|_{A_i})_{S_i \in U})$. Hence, by $K(h, T)$, it follows that the triple $((f^n|_{S_i})_{S_i \in U}, (s^n|_{S_i})_{S_i \in U}, (s^{n+1}|_{S_i})_{S_i \in U}) \in \text{Tr}(U)$. From the proof of Prop. 32 it follows that $(s^n, s^{n+1}) \in \text{Tr}_S(f^n)$, under the assumption that transitions of parallel actions satisfy either **(Disj)** or **(Indep)**. This shows that $K_S(\pi(h), T)$ holds, hence $\pi(h) \in B_T(S)$.

Conversely, let $\bar{h} \in B_T(S)$, where $\bar{h} : T \rightarrow \text{St}(S) \times \text{Pa}(S)$ is defined by $\bar{h}(t) = (s^t, f^t)$, for every $t \in T$. Let $\pi^{-1}(\bar{h}) : T \rightarrow \text{St}(U) \times \text{Pa}(U)$ be defined by $\pi^{-1}(\bar{h})(t) = ((s^t|_{X_i})_{S_i \in U}, (f^t|_{A_i})_{S_i \in U})$. Because $\bar{h} \in B_T(S)$, we know that $K_S(\bar{h}, T)$ holds, hence, if $t, t+1 \in T$, then $(s^t, s^{t+1}) \in \text{Tr}_S(f^t)$. Therefore, by Lemma 26(3), for every $S_i \in U$ we have $(s^t|_{X_i}, s^{t+1}|_{X_i}) \in \text{Tr}_{S_i}(f^t|_{A_i})$, i.e. $((f^t|_{A_i})_{S_i \in U}, (s^t|_{X_i})_{S_i \in U}, (s^{t+1}|_{X_i})_{S_i \in U}) \in \text{Tr}(U)$. Hence, $K(\pi^{-1}(\bar{h}), T)$ is satisfied, i.e. $\pi^{-1}(\bar{h}) \in B_T(U)$.

It is easy to see that the maps π, π^{-1} between $B_T(S)$ and $B_T(U)$ described above are mutually inverse, which proves Prop. 36. \square

Corollary 37 *For every $S_i \in \text{InSys}$, the stalk at S_i is in bijective correspondence with $B_T(S_i)$.*

Proof: Since B_T is a sheaf, by Lemma 20 it follows that the stalk at S_i is in bijective correspondence with $B_T(\downarrow S_i)$. The conclusion now follows immediately from Prop. 36. \square

Theorem 38 ([18]) *Let $B_T(S) = \{h : T \rightarrow \text{St}(S) \times \text{Pa}(S) \mid K_S(h, T)\}$, where $K_S(h, T)$ expresses the fact that for every n , if $n, n+1 \in T$ and $h(n) = (s, f)$, $h(n+1) = (s', f')$ then $(s, s') \in \text{Tr}_S(f)$. Then:*

- For every $T \in \mathcal{T}$, $B_T : \Omega(\text{InSys})^{op} \rightarrow \text{Set}$ is a sheaf.
- For every $S_i \in \text{InSys}$, the stalk at S_i is in bijection with $B_T(S_i)$.
- If the transitions obey **(Disj)** or **(Indep)**, then, for every $U \in \Omega(\text{InSys})$, $B_T(U)$ is in bijection with $B_T(S_U)$, where S_U is the colimit of the diagram defined by U .

Proof: Directly from Prop. 35, Prop. 36, and Cor. 37. \square

7.2 Behavior: Admissible Parallel Actions as Words.

If we ignore the states, the behavior of any system S can be expressed by a subset L_S of the free monoid $\text{Pa}(S)^*$ over the set of possible actions of S , where:

$$L_S = \{f_1 \dots f_n \mid \exists h : \{0, \dots, n\} \rightarrow \text{St}(S) \times \text{Pa}(S), \exists s_i \in \text{St}(S), \text{ s.t.} \\ \forall i \in \{0, \dots, n-1\}, (s_i, s_{i+1}) \in \text{Tr}_S(f_i)\} \subseteq \text{Pa}(S)^*.$$

Consider the family $\{Pa(S_i)^* \mid S_i \in \text{InSys}\}$. If $S_i, S_j \in \text{InSys}$ and $S_i \hookrightarrow S_j$, let $\rho_{S_i}^{S_j} : Pa(S_j) \rightarrow Pa(S_i)$ be the restriction to S_i . The restriction extends to a homomorphism of monoids, $p_i^j : Pa(S_j)^* \rightarrow Pa(S_i)^*$. If there is no risk of confusion, in what follows we will abbreviate $p_i^j(w_j)$ by $w_j|_{S_i}$. Let $M(\text{InSys})$ be defined by:

$$M(\text{InSys}) = \{(w_i)_{S_i \in \text{InSys}} \mid w_i \in Pa(S_i)^* \text{ and } \forall S_i \hookrightarrow S_j, p_i^j(w_j) = w_i\}.$$

It can be seen that $M(\text{InSys})$ is the limit of the diagram $\{Pa(S_i)^* \mid S_i \in \text{InSys}\}$ (with the morphisms p_i^j for every $S_i \hookrightarrow S_j$).

Theorem 39 *Let $M : \Omega(\text{InSys})^{op} \rightarrow \text{Sets}$ be defined as follows:*

Objects: $M(U) = \{(w_i)_{S_i \in U} \mid w_i \in Pa(S_i)^*, w_i|_{S_j} = w_j \text{ for every } S_j \hookrightarrow S_i\}$,
Morphisms: if $\iota : U_1 \subseteq U_2$, $M(\iota) : M(U_2) \rightarrow M(U_1)$ is defined for every
 $(w_i)_{S_i \in U_2}$ by $M(\iota)((w_i)_{S_i \in U_2}) = (w_i)_{S_i \in U_1}$.

Then M is a sheaf of monoids. $M(V)$ is the limit of the diagram $\{Pa(S_i)^ \mid S_i \in V\}$ (with morphisms $p_i^j : Pa(S_j)^* \rightarrow Pa(S_i)^*$ whenever $S_i \hookrightarrow S_j$).*

Proof: Let $U \in \Omega(\text{InSys})$ and $\{U_k \mid k \in K\}$ be a cover for U . Let $\{w_k\}_{k \in K}$ be a family of elements, such that for every $k \in K$, $w_k = (w_k^i)_{S_i \in U_k}$ and for every $k_1, k_2 \in K$, if $S_i \in U_{k_1} \cap U_{k_2}$ then $w_{k_1}^i = w_{k_2}^i$.

We define $w = (w_i)_{S_i \in U}$ as follows: for every $S_i \in U$, $S_i \in U_k$ for some k . Then w_i is defined to be w_k^i . Note that w_i is well defined because of the compatibility of the family $\{w_k\}_{k \in K}$, and $p_{U_k}^U(w) = w_k$ for every $k \in K$. The uniqueness of w follows from the fact that for every $w' = (w'_i)_{S_i \in U}$ such that $p_{U_k}^U(w') = w_k$ for every $k \in K$ we have $w'_i = w_i$ for every $S_i \in U_k$.

The fact that $M(V)$ is the limit of the diagram $\{Pa(S_i)^* \mid S_i \in V\}$ (with the corresponding morphisms) can be checked without difficulty. \square

Remark: Let S be the colimit of the diagram defined by U . The connection between $Pa(S)^*$ and $M(U)$ is rather loose: Let $p : Pa(S)^* \rightarrow M(U)$ be defined by $p(f_1 \dots f_n) = ((f_1 \dots f_n)|_{S_i})_{S_i \in U} \in M(U)$. If we identify the empty action with the empty word ϵ , p may not be injective as can be seen from the following example:

Example 13 Let S_1 and S_2 be as defined in Example 7, where trains are indexed by $I_1 = \{k_1, \dots, n\}$ and $I_2 = \{1, \dots, k_2\}$ and $k_2 < k_1$, with the difference that update is omitted as in Example 4. Let $\text{InSys} = \{S_1, S_2, \emptyset\}$. Let $w_1 = f_1 f_2$ and $w_2 = f_2 f_1$, where

- $f_1^{-1}(1) = \{\text{report}_i \mid i \in I_1\}$ and
- $f_2^{-1}(1) = \{\text{move}_j \mid j \in I_2\}$.

Note that $f_{1|A_1}^{-1}(1) = \{\text{report}_i \mid i \in I_1\}$, $f_{2|A_1}^{-1}(1) = f_{1|A_2}^{-1}(1) = \emptyset$, and $f_{2|A_2}^{-1}(1) = \{\text{move}_j \mid j \in I_2\}$. Thus,

$$\begin{aligned} p(w_1) &= ((f_1 f_2)_{|S_1}, (f_1 f_2)_{|S_2}, (f_1 f_2)_{|\emptyset}) \\ &= ((f_{1|A_1} f_{2|A_1}), (f_{1|A_2} f_{2|A_2}), \epsilon) \\ &= (f_1 \epsilon, \epsilon f_2, \epsilon) = (\epsilon f_1, f_2 \epsilon, \epsilon) \\ &= ((f_{2|A_1} f_{1|A_1}), (f_{2|A_2} f_{1|A_2}), \epsilon) = p(w_2), \end{aligned}$$

but $w_1 \neq w_2$.

The next example shows that $p : Pa(S)^* \rightarrow M(U)$ is not necessarily onto: There may exist compatible families (even if we only consider singleton parallel actions) of sequences of actions that cannot be “glued together” to a sequence of actions on $Pa(S)$. A similar result appears in [13] (in that case, no parallelism is allowed).

Example 14 Let S_1, S_2, S_3 be three systems all having the same language, the same constraints on variables and the same model for the variables, such that

$$\begin{aligned} A_{S_1} &= \{a, b, d\}, & A_{S_2} &= \{b, c, e\}, & A_{S_3} &= \{a, c, f\} \\ C_{S_1} &= \{a \wedge b = 0\} & C_{S_2} &= \{b \wedge c = 0\} & C_{S_3} &= \{a \wedge c = 0\} \end{aligned}$$

Let S be the system obtained by interconnecting S_1, S_2, S_3 . Then:

$$\begin{aligned} A_S &= \{a, b, c, d, e, f\}, \\ C_S &= \{a \wedge b = 0, b \wedge c = 0, a \wedge c = 0\}. \end{aligned}$$

Consider $w_1 = ab \in Pa(S_1)^*$, $w_2 = bc \in Pa(S_2)^*$, $w_3 = ca \in Pa(S_3)^*$. It is easy to see that:

$$\begin{aligned} - p_{12}^1(w_1) &= p_{12}^2(w_2) = b, \\ - p_{23}^2(w_2) &= p_{23}^3(w_3) = c, \\ - p_{13}^1(w_1) &= p_{13}^3(w_3) = a, \end{aligned}$$

but there is no $w \in Pa(S)^*$ such that $w_{|S_i} = w_i, i = 1, 2, 3$.

We investigate therefore other ways of modelling behavior for which tighter links between local and global behavior exist.

7.3 Behavior: Partially Commutative Monoids.

General results on partially commutative monoids. Let $G = (A, D)$ be a graph. (In what follows we will refer to such graphs also as dependency graphs.) We denote by $M(G)$ the quotient A^*/θ , where θ is the congruence on A^* generated by $\{(a_1 a_2, a_2 a_1) \mid (a_1, a_2) \notin D\}$ (a free partially commutative monoid).

Theorem 40 (Corollary 1.4.5 in [3]) *Let $G = (A, D)$ be an undirected graph and $\{G_j = (A_j, D_j) \mid j \in J\}$ be a finite family of subgraphs of G . For $j \in J$ let $\pi_j : M(G) \rightarrow M(G_j)$ be the canonical projection² and $\pi : M(G) \rightarrow \prod_{j \in J} M(G_j)$ be the homomorphism into the direct product defined by $\pi(t) = (\pi_j(t))_{j \in J}$. Then π is injective iff $G = \bigcup_{j \in J} G_j$.*

If $\{M_j \mid j \in J\}$ is a family of non-trivial free partially commutative monoids then $\prod_{j \in J} M_j$ is free partially commutative iff J is finite [3]. If $\{G_j \mid j \in J\}$ is not finite, then – assuming that for every vertex x of G there are finitely many $j \in J$ such that x is a vertex of G_j – there is an injective morphism $M(G) \hookrightarrow \bigoplus_{j \in J} M(G_j)$, where $\bigoplus_{j \in J} M(G_j) = \{(m_j)_{j \in J} \mid m_j \in M(G_j) \text{ for all } j \in J, m_j = \varepsilon \text{ a.e.}\}$ ³ [3], p.27.

The partially-commutative monoid associated with a system. In what follows we assume that the constraints on actions are all of the form $a_i \wedge a_j = 0$ (they state which actions cannot be performed in parallel).

Definition 41 *Let S be a system with the property that the constraints on actions are all of the form $a_i \wedge a_j = 0$. The dependence graph of S is the graph (A_S, D_S) having as set of vertices A_S , and where D_S is defined by $(a_1, a_2) \in D_S$ if $a_1 = a_2$ or $a_1 \wedge a_2 = 0 \in C_S$.*

For every system S with dependence graph (A_S, D_S) we denote by $M(S) = M(A_S, D_S)$ the free partially commutative monoid defined by (A_S, D_S) , i.e. the quotient of A_S^* by the congruence relation generated by $a_1 a_2 = a_2 a_1$ for every $(a_1, a_2) \in (A_S \times A_S) \setminus D_S$. For basic properties of (free) partially commutative monoids we refer e.g. to [3], pp.9-29 and 67-79.

For every $S_i \in \text{InSys} \setminus \{\emptyset\}$, let $M(S_i) = A_{S_i}^* / \theta_i$ (where θ_i is the congruence defined as explained above from $(A_{S_i} \times A_{S_i}) \setminus D_{S_i}$) be the partially commutative monoid associated with the dependence graph of S_i . Let S be the colimit of the diagram defined by InSys . Then $A_S = \bigcup_{S_i \in \text{InSys}} A_i$ and $D_S = \bigcup_{S_i \in \text{InSys}} D_i$. Hence, for every $S_i \in \text{InSys}$ there is a canonical projection $p_i : M(S) \rightarrow M(S_i)$ which is onto. Let $\ker(p_i)$ be the kernel of p_i . Then $M(S_i) \simeq M(S) / \ker(p_i)$.

If $S_i \hookrightarrow S_j$, then we denote the canonical projection by $p_i^j : M(S_j) \rightarrow M(S_i)$, and if $S_i, S_j \in \mathcal{S}$, then $p_{ij}^j : M(S_j) \rightarrow M(S_i \cap S_j)$, and $p_{ij}^i : M(S_i) \rightarrow M(S_i \cap S_j)$ are the canonical mappings. Note that all homomorphisms $p_j^i : M(S_i) \rightarrow M(S_j)$ and $p_{ij}^i : M(S_i) \rightarrow M(S_i \cap S_j)$ are onto. We know that for all $S_j \hookrightarrow S_i$, $p_j^i \circ p_i = p_j$.

Example 15 Consider a family of two systems of trains S_1, S_2 over disjoint sets I_1, I_2 of trains as in Example 7 but with $l < k$. We simplify the description by replacing all actions that need to be executed at the same time with one action.

² π_j is defined by $\pi_j([a_1 \dots a_n]) = [b_1 \dots b_n]$ where if $a_i \in A_j$ then $b_i = a_i$ and if $a_i \in A_j$ then b_i is the empty word.

³ a.e. means *almost everywhere*

The system S_i ($i \in \{1, 2\}$) obtained this way has two actions update_i and move_i . The constraints are $C_i = \{\text{update}_i \wedge \text{move}_i = 0\}$. Thus $\theta_i = \text{id}$, so $M(S_i) = A_{S_i}^*$. Let S be the system obtained by the interconnection of S_1 and S_2 .

$$\begin{aligned} A_S &= \{\text{update}_1, \text{update}_2, \text{move}_1, \text{move}_2\} \text{ and } C_S = C_1 \cup C_2. \\ D_S &= \{(\text{update}_1, \text{update}_1), (\text{update}_2, \text{update}_2), (\text{move}_2, \text{move}_2), (\text{move}_1, \text{move}_1), \\ &\quad (\text{update}_1, \text{move}_1), (\text{move}_1, \text{update}_1), (\text{update}_2, \text{move}_2), (\text{move}_2, \text{update}_2)\} \\ (A_S \times A_S) \setminus D_S &= \{(\text{update}_1, \text{update}_2), (\text{update}_2, \text{update}_1), (\text{update}_1, \text{move}_2), \\ &\quad (\text{move}_2, \text{update}_1), (\text{move}_1, \text{update}_2), (\text{update}_2, \text{move}_1), \\ &\quad (\text{move}_1, \text{move}_2), (\text{move}_2, \text{move}_1)\} \end{aligned}$$

Thus, $M(S) = A_S^*/\theta$, where θ is the congruence generated by $(A_S \times A_S) \setminus D_S$.

Applying a method due to [2] (cf. Section 3.2) – where sheaves of algebras are constructed, whose stalks are quotients of a given algebra – we deduce for partially commutative monoids results similar to those given in [13] for monoids. The results are similar to results on Petri Nets and Mazurkiewicz traces presented in [3].

Let (F, f, InSys) be defined by $F = \prod_{S_i \in \text{InSys}} M(S_i)$, and $f : F \rightarrow \text{InSys}$ be the natural projection. Assume that a subbasis for the topology on F is $\mathcal{SB} = \{[m](U) \mid U \in \Omega(\text{InSys}), m \in M(S)\}$, where $[m](U) = \{p_i(m) \mid i \in U\}$.

We first show that $\Omega(\text{InSys})$ has the property that for every $m_1, m_2 \in M(S)$, if $p_i(m_1) = p_i(m_2)$ then there exists an open neighborhood U of S_i in $\Omega(\text{InSys})$ such that for every $S_j \in U$, $p_j(m_1) = p_j(m_2)$ (i.e. it is an S-topology).

Lemma 42 $\Omega(\text{InSys})$ is a S-topology (cf. Definition 14).

Proof: We show that for every $m_1, m_2 \in M(S)$, if $p_i(m_1) = p_i(m_2)$ then there exists an open neighborhood U of S_i in $\Omega(\text{InSys})$ s.t. for every $S_j \in U$, $p_j(m_1) = p_j(m_2)$. Let $m_1, m_2 \in M(S)$ with $p_i(m_1) = p_i(m_2)$. Let $U = \downarrow S_i = \{S_j \in \text{InSys} \mid S_j \hookrightarrow S_i\}$. $U \in \Omega(\text{InSys})$ and $p_j(m_1) = p_j^i(p_i(m_1)) = p_j^i(p_i(m_2)) = p_j(m_2)$ for every $S_j \in U$. \square

Let $\alpha : M(S) \rightarrow \Gamma(I, F_A)$ be defined by $\alpha(m) = ([m]_{\theta_i})_{i \in I}$. Since $\Omega(\text{InSys})$ is an S-topology, by Theorem 13 and Corollary 15 in Section 3.2 we have:

- (1) (F, f, InSys) is a sheaf of algebras,
- (2) The stalk at $S_i \in \text{InSys}$ is isomorphic to $M(S_i)$,
- (3) In $M(S) \xrightarrow{\alpha} \Gamma(\text{InSys}, F) \leq \prod_{S_i \in \text{InSys}} M(S_i) \xrightarrow{\pi_i} M(S_i)$
 - (3.i) $\pi_i \circ \alpha$ is an epimorphism,
 - (3.ii) $M(S)$ is a subdirect product of $\{M(S_i)\}_{S_i \in \text{InSys}}$ iff α is a monomorphism.

Lemma 43 Let $s : \text{InSys} \rightarrow \prod_{S_i \in \text{InSys}} M(S_i)$ be such that $s(S_i) \in M(S_i)$ for every $S_i \in \text{InSys}$. Let $m \in M(S)$ and $U \in \Omega(\text{InSys})$. Then $S_i \in s^{-1}([m](U))$ if and only if $S_i \in U$ and $s(S_i) = p_i(m)$.

Proof: Note that $s^{-1}([m](U)) = \{S_i \in \text{InSys} \mid s(S_i) \in [m](U)\} = \{S_i \in \text{InSys} \mid s(S_i) \in \{p_j(m) \mid S_j \in U\}\}$. We first prove the direct implication. Assume that $S_i \in s^{-1}([m](U))$. Then $s(S_i) = p_j(m)$ for some $S_j \in U$. Since $f \circ s(S_i) = S_i$, it follows that $S_i = f(s(S_i)) = f(p_j(m)) = S_j$, hence $S_i \in U$ and $s(S_i) = p_i(m)$. To prove the converse, assume that $S_i \in U$ and $s(S_i) = p_i(m)$. Then $s(S_i) \in \{p_j(m) \mid S_j \in U\}$, hence $S_i \in s^{-1}([m](U))$. \square

Lemma 44 *Let τ be the topology on $F = \coprod_{S_i \in \text{InSys}} M(S_i)$ generated by*

$$\mathcal{SB} = \{[m](U) \mid U \in \Omega(\text{InSys}), m \in M(S)\}$$

as a subbasis. Then $s : \text{InSys} \rightarrow \coprod_{S_i \in \text{InSys}} M(S_i)$ such that for every $S_i \in \text{InSys}$, $s(S_i) \in M(S_i)$ is continuous if and only if for every $S_i, S_j \in \text{InSys}$ such that $S_j \hookrightarrow S_i$, $p_j^i(s(S_i)) = s(S_j)$.

Proof: Since \mathcal{SB} is a subbasis for the topology on $F = \coprod_{S_i \in \text{InSys}} M(S_i)$, the map $s : \text{InSys} \rightarrow \coprod_{S_i \in \text{InSys}} M(S_i)$ is continuous iff for every $[m](U) \in \mathcal{SB}$, $s^{-1}([m](U)) \in \Omega(\text{InSys})$. We first prove the direct implication. Assume that $s : \text{InSys} \rightarrow \coprod_{S_i \in \text{InSys}} M(S_i)$ is continuous. Let $S_i, S_j \in \text{InSys}$ be such that $S_j \hookrightarrow S_i$. We prove that $p_j^i(s(S_i)) = s(S_j)$. Let $U = \downarrow S_i \in \Omega(\text{InSys})$ and let $m \in M(S)$ be such that $p_i(m) = s(S_i)$ (the existence of m is ensured by the fact that $p_i : M(S) \rightarrow M(S_i)$ is onto). From the continuity of s we know that $s^{-1}([m](\downarrow S_i)) \in \Omega(\text{InSys})$. Obviously, $S_i \in s^{-1}([m](\downarrow S_i))$. Therefore, since $S_j \hookrightarrow S_i$, $S_j \in s^{-1}([m](\downarrow S_i))$, hence, by Lemma 43, $s(S_j) = p_j(m)$. Therefore, $s(S_j) = p_j(m) = p_j^i(p_i(m)) = p_j^i(s(S_i))$.

Conversely, assume that for every $S_i, S_j \in \text{InSys}$ such that $S_j \hookrightarrow S_i$ it holds that $p_j^i(s(S_i)) = s(S_j)$. We prove that s is continuous. Let $[m](U) \in \mathcal{SB}$, where $m \in M(S)$ and $U \in \Omega(\text{InSys})$. We prove that $s^{-1}([m](U)) \in \Omega(\text{InSys})$. Let $S_i \in s^{-1}([m](U))$. Then $S_i \in U$ and $s(S_i) = p_i(m)$. Let $S_j \hookrightarrow S_i$. Then $S_j \in U$ and by the hypothesis, $s(S_j) = p_j^i(s(S_i)) = p_j^i(p_i(m)) = p_j(m)$. Thus, $S_j \in s^{-1}([m](U))$. Therefore $s^{-1}([m](U)) \in \Omega(\text{InSys})$. \square

Lemma 45 *The set $\Gamma(\text{InSys}, F)$ of global sections of F has the form*

$$\Gamma(\text{InSys}, F) = \{(m_i)_{S_i \in \text{InSys}} \mid m_i \in M(S_i) \text{ and } \forall S_j \hookrightarrow S_i \in \text{InSys}, p_j^i(m_i) = m_j\}.$$

Proof: We know that $\Gamma(\text{InSys}, F) = \{s : \text{InSys} \rightarrow \coprod_{S_i \in \text{InSys}} M(S_i) \mid s \text{ continuous and } s(S_i) \in M(S_i), \forall S_i \in \text{InSys}\}$. (The elements of $\Gamma(\text{InSys}, F)$ are tuples $(s(S_i))_{S_i \in \text{InSys}}$.) Let first $s \in \Gamma(\text{InSys}, F)$. Then s is continuous and, by Lemma 44, for all $S_i, S_j \in \text{InSys}$ with $S_j \hookrightarrow S_i$, $p_j^i(s(S_i)) = s(S_j)$. Conversely, let $(m_i)_{S_i \in \text{InSys}}$ be such that for every $S_i, S_j \in \text{InSys}$, $m_i \in M(S_i)$ if $S_j \hookrightarrow S_i$ then $p_j^i(m_i) = m_j$. Let $s : \text{InSys} \rightarrow \coprod_{S_i \in \text{InSys}} M(S_i)$ be defined by $s(S_i) = m_i$ for every $S_i \in \text{InSys}$. Then, whenever $S_j \hookrightarrow S_i \in \text{InSys}$, $p_j^i(s(S_i)) = s(S_j)$ and, by Lemma 44, s is continuous. \square

Theorem 46 Let (F, f, InSys) be defined as above. Then (F, f, InSys) is a sheaf space of algebras. The stalk at $S_i \in \text{InSys}$ is isomorphic to $M(S_i)$. The set of global sections is

$$\Gamma(\text{InSys}, F) = \{(m_i)_{S_i \in \text{InSys}} \mid m_i \in M(S_i), \text{ and } \forall S_i \hookrightarrow S_j, p_i^j(m_j) = m_i\}.$$

Additionally the following hold:

- (1) If InSys is finite, then
 - (i) $M(S) \hookrightarrow \Gamma(\text{InSys}, F) \leq \prod_{S_i \in \text{InSys}} M(S_i) \xrightarrow{\pi_i} M(S_i)$ is a subdirect product.
 - (ii) The embedding $M(S) \hookrightarrow \Gamma(\text{InSys}, F)$ is an isomorphism iff every chordless cycle in the dependence graph G_S of S is a cycle in a subgraph G_{S_i} for some $S_i \in \text{InSys}$.
- (2) If InSys is infinite, and if for every $a \in A_S$ there are at most finitely many $S_i \in \text{InSys}$ with $a \in A_i$, then there is an injective morphism $M(S) \rightarrow \bigoplus_{S_i} M(S_i)$, where $\bigoplus_{S_i} M(S_i) = \{(w_i)_{i \in I} \mid w_i \in M(S_i), w_i = \varepsilon \text{ a.e.}\}$ is the weak product of the family $\{M(S_i)\}_{S_i \in \text{InSys}}$.

Proof: The form of $\Gamma(\text{InSys}, F)$ follows from Lemma 45. (1)(i) and (2) are a consequence of Theorem 40 and the subsequent comments at the beginning of Section 7.3. (1)(ii) is a direct consequence of Theorem 3.3.2 in [3]. \square

Example 16 First consider the family of systems in Example 15. The dependency graph of S , $G_S = (A_S, D_S)$ contains the following non-trivial chordless cycles:

1. $(\text{update}_1, \text{move}_1, \text{update}_1)$ and $(\text{move}_1, \text{update}_1, \text{move}_1)$ (all cycles in G_{S_1})
2. $(\text{update}_2, \text{move}_2, \text{update}_2)$ and $(\text{move}_2, \text{update}_2, \text{move}_2)$ (all cycles in G_{S_2}).

Thus, in this case the embedding in Theorem 46(1)(ii) is an isomorphism.

Example 17 Consider the systems in Example 14. The dependency graphs are:

- $G_{S_1} = (A_1, D_1)$, with $D_1 = \{(a, a), (b, b), (d, d), (a, b), (b, a)\}$,
- $G_{S_2} = (A_2, D_2)$, with $D_2 = \{(b, b), (c, c), (e, e), (b, c), (c, b)\}$,
- $G_{S_3} = (A_3, D_3)$, with $D_3 = \{(a, a), (c, c), (f, f), (a, c), (c, a)\}$.

$G_S = (A_1 \cup A_2 \cup A_3, D_1 \cup D_2 \cup D_3)$ contains the chordless cycle (a, b, c, a) which is not contained in any of the subgraphs $G_{S_i}, i \in \{1, 2, 3\}$. Thus, the embedding in Theorem 46(1)(ii) is not an isomorphism.

8 Other concepts and their sheaf semantics

We now analyze other concepts which are important when describing the properties of systems and their sheaf semantics.

8.1 Time

We assume that time is discrete, hence it can be modeled by the set \mathbb{N} of natural numbers.

One possibility for expressing time internally in the category $\mathbf{Sh}(\mathbf{InSys})$ is to model time by the sheafification \mathbb{N} of the constant presheaf $\mathcal{N} : \Omega(\mathbf{InSys})^{\text{op}} \rightarrow \mathbf{Set}$ (defined for every U by $\mathcal{N}(U) = \mathbb{N}$), which can be constructed as follows:

- (1) Let $\mathcal{N}^+ : \Omega(\mathbf{InSys})^{\text{op}} \rightarrow \mathbf{Sets}$, defined by $\mathcal{N}^+(U) = \mathbb{N}$ if $U \neq \emptyset$ and $\mathcal{N}^+(\emptyset) = 1$ (for the empty cover there is exactly one matching family; the empty one).
- (2) Let $\mathbb{N} = (\mathcal{N}^+)^+ : \Omega(\mathbf{InSys})^{\text{op}} \rightarrow \mathbf{Sets}$. $n_j \in \mathcal{N}(U_j)$ for some open covering $\{U_j \mid j \in J\}$ of U , which match ($n_{j_1} = n_{j_2}$) whenever the overlap $U_{j_1} \cap U_{j_2}$ is nonempty.

Thus, these elements “glue” together to give a function $n : U \rightarrow \mathbb{N}$, with the property that every point of U has some open neighborhood on which the function is constant.

Therefore (cf. also [12]), for every $U \in \Omega(\mathbf{InSys})$, $\mathbb{N}(U) = \{i : U \rightarrow \mathbb{N} \mid f \text{ locally constant}^4\}$. There exist $\mathbf{Sh}(\mathbf{InSys})$ -arrows $1 \xrightarrow{0} \mathbb{N} \xrightarrow{s} \mathbb{N}$; the sheaf \mathbb{N} is the natural number object in $\mathbf{Sh}(\mathbf{InSys})$.

Theorem 47 *For every $U \in \Omega(\mathbf{InSys})$ there is a bijection between $\mathbb{N}(U)$ and $\{(n_i)_{S_i \in U} \mid \text{if } S_i \hookrightarrow S_j \text{ then } n_i = n_j\}$. For every $S_i \in \mathbf{InSys}$, the stalk of \mathbb{N} at S_i is in bijection with the set \mathbb{N} of natural numbers.*

Proof: Let $p : \mathbb{N}(U) \rightarrow \{(n_i)_{S_i \in U} \mid \text{if } S_i \hookrightarrow S_j \text{ then } n_i = n_j\}$ be defined by $p(n) = (n(S_i))_{S_i \in U}$. Since n is locally constant and every open neighborhood of $S_i \in U$ contains $\downarrow S_i$, it follows that n is constant on $\downarrow S_i$. This proves that for every $S_i \hookrightarrow S_j$, $n(S_j) = n(S_i)$, i.e. p is well-defined. The inverse of p associates with every $(n_i)_{S_i \in U}$ the map $q((n_i)_{S_i \in U}) : U \rightarrow \mathbb{N}$, $q((n_i)_{S_i \in U})(S_i) = n_i$. It is easy to see that $q((n_i)_{S_i \in U})$ is locally constant (for every $S_i \in U$, the required neighborhood is $\downarrow S_i$).

The second part follows from Lemma 20 and the fact that every locally constant map on $\downarrow S_i$ is constant. \square

8.2 Other constructions

Various other sheaves and natural transformations can be defined by using standard categorical constructions in $\mathbf{Sh}(\mathbf{InSys})$. We can e.g. define a natural transformation $\mathbf{B}_{\mathbb{N}} \times \mathbb{N} \xrightarrow{\mathbf{a}} \mathbf{St} \times \mathbf{Pa}$ whose components $\mathbf{B}_{\mathbb{N}}(U) \times \mathbb{N}(U) \xrightarrow{\mathbf{a}_U} \mathbf{St}(U) \times \mathbf{Pa}(U)$ are defined by $\mathbf{a}_U(h, (n_i)_{S_i \in U}) = ((s_i^i)_{S_i \in U}, (f_i^i)_{S_i \in U})$, for every $U \in \Omega(\mathbf{InSys})$, where for every $S_i \in U$, $h(n_i) = ((s_j^i)_{S_j \in U}, (f_j^i)_{S_j \in U})$.⁵

⁴ $f:U \rightarrow X$ is locally constant if $\forall x \in U$ there is an open neighborhood $U_1 \subseteq U$ of x on which f is constant. This means that ‘local clocks’ of the systems in U synchronize for systems sharing common subsystems.

⁵ The map \mathbf{a}_U has as arguments a behaviour along \mathbb{N} of the family of systems in U , $h \in \mathbf{B}_{\mathbb{N}}(U)$, and a tuple consisting of ‘local clocks’ of the systems in U which synchronize

Theorem 48 Let $\mathbf{B}_{\mathbb{N}} \times \mathbb{N} \xrightarrow{\mathbf{a}} \mathbf{St} \times \mathbf{Pa}$ be the natural transformation with components $\mathbf{B}_{\mathbb{N}}(U) \times \mathbb{N}(U) \xrightarrow{\mathbf{a}_U} \mathbf{St}(U) \times \mathbf{Pa}(U)$ defined by

$$\mathbf{a}_U(h, (n_i)_{S_i \in U}) = ((s_i^i)_{S_i \in U}, (f_i^i)_{S_i \in U}),$$

where for every $S_i \in U$, $h(n_i) = ((s_j^i)_{S_j \in U}, (f_j^i)_{S_j \in U})$. Then the restriction to the stalks is (up to isomorphism) the map $B_T(S_i) \times \mathbb{N} \xrightarrow{\mathbf{a}^{S_i}} St(S_i) \times Pa(S_i)$, defined by $\mathbf{a}_{S_i}(h, n) = h(n)$.

Proof: Follows immediately from the way the stalk functors are defined on morphisms, and from the fact that, by Theorems 34, 38, and 47, $F_{S_i} \simeq F(\downarrow S_i) \simeq F(S_i)$ for $F \in \{\mathbf{St}, \mathbf{Pa}, \mathbf{B}_T\}$ and $\mathbb{N}_{S_i} \simeq \mathbb{N}(\downarrow S_i) \simeq \mathbb{N}$. \square

A similar result holds also for global sections.

Note: The empty system \emptyset can be excluded from considerations, as follows. Let $\Omega_1(\mathbf{InSys})$ be the family of all subsets of $\mathbf{InSys} \setminus \emptyset$ closed under t.c. subsystems. (If no element in \mathbf{InSys} is the colimit of other elements in \mathbf{InSys} , then $\Omega_1(\mathbf{InSys})$ is the free frame freely generated by \mathbf{InSys} together with the constraint that the empty family of systems covers the empty system.) All the considerations above remain valid when $\Omega(\mathbf{InSys})$ is replaced by $\Omega_1(\mathbf{InSys})$. The space $(\mathbf{InSys} \setminus \emptyset, \Omega_1(\mathbf{InSys}))$ is totally disconnected if for every $S_1, S_2 \in \mathbf{InSys}$, their largest common t.c. subsystem $S_1 \wedge S_2$ is empty; the space is compact if additionally \mathbf{InSys} is finite. In this situation a larger class of axioms is preserved by the global section functor (uniqueness in existential quantification is not required, cf. e.g. [9], Ch. V.1.12). Then, the definition of time as a sheaf \mathbb{N} expresses the fact that independent systems may have independent clocks.

9 Geometric logic and properties of systems

We provide interpretations for properties of systems (i.e. statements about states, actions, behavior) both concretely (in the category of sets) and in a category of sheaves, and establish links between the set-theoretical (both for individual systems and for their interconnections) and the sheaf-theoretical interpretation. These links are then used to prove preservation of truth when interconnecting systems.

9.1 Many-sorted first order languages and their interpretation in $\mathbf{Sh}(I)$

Let \mathcal{L} be a many-sorted first-order language consisting of a collection of sorts and collections of function and relation symbols. Terms and atomic formulae from \mathcal{L} are defined in the standard way; compound formulae are constructed by using the connectives $\vee, \wedge, \Rightarrow, \neg$ and the quantifiers \exists, \forall , for every sort X . An *interpretation* M of \mathcal{L} in $\mathbf{Sh}(I)$ is constructed by associating:

on systems sharing common subsystems. \mathbf{a}_U returns the pair $((s_i^i)_{S_i \in U}, (f_i^i)_{S_i \in U})$ where (s_i^i, f_i^i) is the pair state/parallel action in the behavior corresponding to the system S_i in U , at the time point indicated by the local clock n_i of S_i .

- a sheaf X^M on I to every sort X ,
- a subsheaf $R^M \subseteq X_1^M \times \cdots \times X_n^M$ to every relation symbol R of arity $X_1 \times \cdots \times X_n$,
- an arrow $f^M : X_1^M \times \cdots \times X_n^M \rightarrow Y^M$ in $\text{Sh}(I)$ to every function symbol f with arity $X_1 \times \cdots \times X_n \rightarrow Y$.

Each term $t(x_1, \dots, x_n)$ of sort Y is (inductively) interpreted as an arrow $t^M : X_1^M \times \cdots \times X_n^M \rightarrow Y^M$; and every formula $\phi(x_1, \dots, x_n)$ with free variables $FV(\phi) \subseteq \{x_1, \dots, x_n\}$, where x_i is of sort X_i , gives rise to a subsheaf $\{(x_1, \dots, x_n) \mid \phi(x_1, \dots, x_n)\}^M \subseteq X_1^M \times \cdots \times X_n^M$. For details we refer to [12], Ch. X.

Definition 49 *A geometric formula is a formula built from atomic formulae by using only the connectives \vee and \wedge and the quantifier \exists . A geometric axiom is a formula of the form $(\forall x_1, \dots, x_n)(\phi \Rightarrow \psi)$ where ϕ and ψ are geometric formulae.*

Let \mathbb{T} be a theory in the language \mathcal{L} . A variable in a geometric formula is called \mathbb{T} -provably unique if its value in every model of \mathbb{T} is uniquely determined by the values of the remaining free variables.

A cartesian formula w.r.t. \mathbb{T} is a formula constructed from atomic formulae using only the connective \wedge and the quantifier \exists over \mathbb{T} -provably unique variables. A cartesian axiom w.r.t. \mathbb{T} is a formula of the form $(\forall x)(\phi(x) \Rightarrow \psi(x))$ where ϕ and ψ are cartesian formulae w.r.t. \mathbb{T} . A cartesian theory is a theory whose axioms can be ordered such that each is cartesian w.r.t. the preceding ones.

A geometric axiom $(\forall x_1 \dots x_n)(\phi \Rightarrow \psi)$ is satisfied in an interpretation M in $\text{Sh}(I)$ if $\{(x_1, \dots, x_n) \mid \phi\}^M$ is a subobject of $\{(x_1, \dots, x_n) \mid \psi\}^M$ in $\text{Sh}(I)$.

9.2 Stalk functors, global section functors

Stalk functors. For every $S_i \in \text{InSys}$ let $f_i : \{*\} \rightarrow \text{InSys}$ be defined by $f_i(*) = S_i$. The inverse image functor corresponding to f_i , the stalk functor $\text{Stalk}_{S_i} = f_i^* : \text{Sh}(\text{InSys}) \rightarrow \text{Set}$, associates to every sheaf $F \in \text{Sh}(\text{InSys})$ the stalk at S_i , F_{S_i} . For all $S_i \in \text{InSys}$, f_i^* preserves the validity of geometric axioms. The stalk functors f_i^* are collectively faithful, so they reflect the validity of geometric axioms.

Global section functor. Consider the unique map $g : \text{InSys} \rightarrow \{*\}$. The direct image functor, $g_* : \text{Sh}(\text{InSys}) \rightarrow \text{Set}$, is the global section functor $g_*(F) = F(\text{InSys})$ for every $F \in \text{Sh}(\text{InSys})$. Thus, the global section functor preserves the interpretation of every cartesian axiom.

9.3 Geometric logic and reasoning about complex systems

Let \mathcal{L} be a fixed many-sorted language including at least sorts like $\text{st}(\text{ate})$, $\text{pa}(\text{allel-action})$, $\text{b}(\text{ehavior})$, $\text{t}(\text{ime})$; constants like $s_0 : \text{st}$ (initial state), $0 : \text{t}$ (initial moment of time); function symbols like $\text{appl} : \text{b} \times \text{t} \rightarrow \text{st} \times \text{pa}$, $\text{p}_1 : \text{st} \times \text{pa} \rightarrow \text{st}$, $\text{p}_2 : \text{st} \times \text{pa} \rightarrow \text{pa}$; relation symbols like $\text{tr}(\text{ansition}) \subseteq \text{pa} \times \text{st} \times \text{st}$, $=_X \subseteq X \times X$

for every sort X , etc. Let M be an interpretation of \mathcal{L} in $\text{Sh}(\text{InSys})$ such that $\text{st}^M = \text{St}$, $\text{pa}^M = \text{Pa}$, $\text{b}^M = \text{B}_{\mathbb{N}}$, $\text{t} = \mathbb{N}$, $\text{appl}^M = \text{a}$, $\text{p}_1^M = \pi_1$, $\text{p}_2^M = \pi_2$ (the canonical projections), $\text{tr}^M = \text{Tr}$. For every sort X , we interpret $=_X: X \times X \rightarrow \Omega$ as usual.

Theorem 50 $\text{Sh}(\text{InSys})$ satisfies a geometric axiom in the interpretation M if and only if Set satisfies it in all interpretations $f_i^*(M)$. If $\text{Sh}(\text{InSys})$ satisfies a cartesian axiom, this is also true in Set in the interpretation $g_*(M)$ ($f_i^*(M)$ and $g_*(M)$ interpret a sort X as $f_i^*(X^M)$ resp. $g_*(X^M)$).

From Theorems 34 and 38 we know that for every $S_i \in \text{InSys}$, $f_i^*(\text{St}) = \text{St}_{S_i} \simeq \text{St}(S_i)$ and $f_i^*(\text{Pa}) = \text{Pa}_{S_i} \simeq \text{Pa}(S_i)$; if S is the system obtained by interconnecting all elements in InSys , $g_*(\text{St}) = \text{St}(\text{InSys}) \simeq \text{St}(S)$, and $g_*(\text{Pa}) = \text{Pa}(\text{InSys}) \simeq \text{Pa}(S)$. The same holds for Tr and B_T . Moreover, $f_i^*(\mathbb{N}) = \mathbb{N}$, $g_*(\mathbb{N}) = \mathbb{N}(\text{InSys})$, and, by Theorem 48, $f_i^*(\text{appl}) = \text{a}_{S_i}: \text{B}_{\mathbb{N}}(S_i) \times \mathbb{N} \rightarrow \text{St}(S_i) \times \text{Pa}(S_i)$. Hence, statements about states, actions and transitions in $\text{Sh}(\text{InSys})$ are translated by f_i^* (resp. g_*) to corresponding statements about states, actions and transitions in S_i (resp. S).

We illustrate the ideas above by several classes of properties of systems (adapted from [11]) which we express in the language \mathcal{L} . For instance, if h is a possible behavior and j a moment in time, then $h(j)$ can be expressed in \mathcal{L} by $\text{appl}(h, j)$; the state of h at j can be expressed by $\text{s}(h, j)$, where $\text{s} = \text{p}_1 \circ \text{appl}: \text{b} \times \text{t} \xrightarrow{\text{appl}} \text{st} \times \text{pa} \xrightarrow{\text{p}_1} \text{st}$.

(a) **Safety properties** are of the form

$$(\forall h : \text{b})(\forall j : \text{t})(P(\text{s}(h, 0)) \Rightarrow Q(\text{s}(h, j))),$$

where P and Q are formulae in \mathcal{L} . As examples we mention:

partial correctness:

$$(\forall h : \text{b})(\forall j : \text{t})[(P(\text{s}(h, 0)) \wedge \text{Final}(\text{s}(h, j))) \Rightarrow Q(\text{s}(h, j))];$$

global invariance of Q :

$$(\forall h : \text{b})(\forall j : \text{t})[P(\text{s}(h, 0)) \Rightarrow Q(\text{s}(h, j))].$$

(b) **Liveness properties** have the form

$$(\forall h : \text{b})[P(\text{s}(h, 0)) \Rightarrow (\exists j : \text{t})Q(\text{s}(h, j))].$$

With s_0 denoting the initial and s_f a final state, examples are:

total correctness and termination:

$$(\forall h : \text{b})[P(\text{s}(h, 0)) \Rightarrow (\exists j : \text{t})(\text{Final}(\text{s}(h, j)) \wedge Q(\text{s}(h, j)))];$$

accessibility:

$$(\forall h : \text{b})[(\text{s}(h, 0) = s_0) \Rightarrow (\exists j : \text{t})(\text{s}(h, j) = s_f)].$$

(c) **Precedence properties** have the form:

$$(\forall h : \mathbf{b})(\forall j : \mathbf{t})[(P(\mathbf{s}(h, 0)) \wedge A(\mathbf{s}(h, j))) \Rightarrow Q(\mathbf{s}(h, j))].$$

Theorem 51 ([18]) *Assume that the following conditions are fulfilled:*

- (1) *The final states form a subsheaf $\mathbf{St}_f \subseteq \mathbf{St}$ interpreting a sort \mathbf{st}_f of \mathcal{L} . (This happens e.g. if in the definition of a system final states are specified by additional constraints, and in defining colimits this information is also used.)*
- (2) *The properties P, Q, A can be expressed in \mathcal{L} (using the sorts, constants, function and relation symbols mentioned at the beginning of Section 9), and can be interpreted in $\mathbf{Sh}(\mathbf{InSys})$ and also in \mathbf{Set} (to express, for every S_i in \mathbf{InSys} , the corresponding property of S_i , or S).*

The truth of formulae describing safety, liveness and precedence properties (as in (a),(b),(c) above) is preserved under inverse image functors if in the definitions of the property P (c.q. Q, A) only conjunction, disjunction and existential quantification occur. The truth of these formulae is additionally preserved by direct image functors if only conjunction and unique existential quantification occur in them.

10 Examples

We illustrate the results we obtained on two examples.

10.1 Example 1: Safety of train system

Consider the example in Section 7: Let $k \leq l \in \{1, \dots, n\}$, $I_1 = \{k, \dots, n\}$, $I_2 = \{1, \dots, l\}$, and $I_{12} = \{k, \dots, l\}$. Let $\mathbf{InSys} = \{S_1, S_2, S_{12}\}$ be the family consisting of the subsystems of S described in Section 1 corresponding to the sets of trains with indices in I_1, I_2 and I_{12} . Let Γ_s^j , $j \in \{1, 2, 12\}$ be the following constraints encoding collision freeness of S_j (where \Rightarrow denotes logical implication):

$$\Gamma_s^j = \{\text{succ}(\text{TrainIndex}_i) = \text{TrainIndex}_k \Rightarrow \text{ActualPos}_i < \text{ActualPos}_k - L \mid i, k \in I_j\}.$$

For every $S_j \in \{1, 2, 12\}$ let $\text{SafeSt}(S_j) = \{s : X_j \rightarrow M_j \mid s \models \Gamma_j \cup \Gamma_s^j\}$ be the set of safe states of S_j ⁶. Let $\text{SafeState} : \Omega(\mathbf{InSys}) \rightarrow \mathbf{Sets}$ be defined on objects by $\text{SafeState}(U) = \{(s_j)_{S_j \in U} \mid s_j \in \text{SafeSt}(S_j), \text{ and } s_j|_{X_i} = s_i \text{ whenever } S_i \hookrightarrow S_j\}$, and on morphisms by restriction. We can define a set of similar constraints Γ_s and a similar set of safe states $\text{SafeSt}(S)$ for the system S , where:

$$\Gamma_s = \{\text{succ}(\text{TrainIndex}_i) = \text{TrainIndex}_k \Rightarrow \text{ActualPos}_i < \text{ActualPos}_k - L \mid i, k \in \{1, \dots, n\}\}.$$

If $I_1 \cap I_2 \neq \emptyset$ then $\Gamma_s^1 \cup \Gamma_s^2 = \Gamma_s$ ⁷. Analogously to Theorem 34 we can show:

⁶ We denote by Γ_j the restriction of Γ (cf. Definition 1) to X_j

⁷ Note that if $I_1 \cap I_2 = \emptyset$ then some of the constraints of Γ_s cannot be deduced from Γ_s^1 and Γ_s^2

Theorem 52 *The following hold:*

1. *SafeState is a sheaf. Moreover, SafeState is a subsheaf of St.*
2. *For each $S_i \in \text{InSys}$, the stalk of SafeState at S_i is in bijection with $\text{SafeSt}(S_i)$.*
3. *$\text{SafeState}(\text{InSys})$ is in bijection with $\text{SafeSt}(S)$.*

Collision freeness can be expressed as follows:

$$\text{CollFree} \quad (\forall h : \mathbf{b})(\forall j : \mathbf{t}) [\text{SafeState}(s(h, 0)) \Rightarrow \text{SafeState}(s(h, j))].$$

This formula contains only atomic formulae and the implication symbol. Therefore, by Theorem 51, its truth is preserved both under inverse image functors and under direct image functors, and it is reflected by the stalk functors:

- Assume that S_1, S_2, S_{12} satisfy CollFree . Then for all $h \in B_{\mathbb{N}}(S_j)$, $t \in \mathbb{N}$, if $\pi_1(h(0)) \in \text{SafeSt}(S_j)$ then $\pi_1(h(t)) \in \text{SafeSt}(S_j)$. Due to the form of the formula CollFree , its truth is reflected by the stalk functors $f_j^* : \text{Sh}(\text{InSys}) \rightarrow \text{Set}$. It therefore follows that $\text{Sh}(\text{InSys})$ satisfies, internally, the formula CollFree .
- The truth of CollFree is preserved by the global section functor $g_* : \text{Sh}(\text{InSys}) \rightarrow \text{Set}$, defined by $g(F) = F(\text{InSys})$. Therefore, (in Set) the following holds:

$$\forall h \in B_{\mathbb{N}}(\text{InSys}), \forall t \in \mathbb{N}(\text{InSys}) [\pi_1(h(0)) \in \text{SafeState}(\text{InSys}) \Rightarrow \pi_1(h(t)) \in \text{SafeState}(\text{InSys})]$$

As, by Theorems 52 and 38, $\text{SafeState}(\text{InSys})$ is in bijective correspondence with $\text{SafeSt}(S)$ and $B_{\mathbb{N}}(\text{InSys})$ is in bijective correspondence with $B_{\mathbb{N}}(S)$, we obtain:

$$\forall h \in B_{\mathbb{N}}(S), \forall t \in \mathbb{N}, \text{ if } \pi_1(h(0)) \in \text{SafeSt}(S) \text{ then } \pi_1(h(t)) \in \text{SafeSt}(S).$$

This states that the system S is collision free.

Corollary 53 *Consider a family of consecutive trains on a linear track without loops. Assume that each train i controls both its position and the position of its predecessor, and accordingly determines its movement mode. We obtain a family $\{S_i \mid i \in \{2, \dots, n\}\}$ of systems consisting of two successor trains each (each defined as in Example 1 for $n = 2$). Let U consist of this family of systems together*

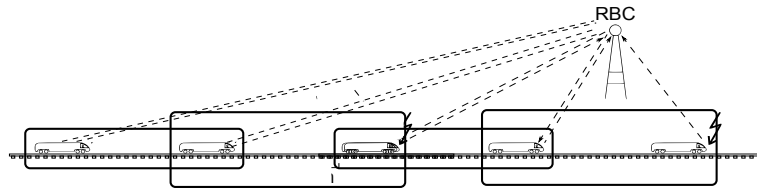


Fig. 7. Covering with a family of systems consisting of two successor trains

with their intersections. The colimit of this family is the system S described in Example 6. By Theorem 51, if collision freeness can be guaranteed for all the systems in U , then the system S is collision free.

For suitably chosen minSpeed , maxSpeed and update interval Δt it can be proved that the 2-train systems are collision free (for an automatic proof ideas from [8] can be used). Therefore, the n -train system in Example 1 can be proved to be collision free for these values.

Remark: The condition that the systems consist of successive trains and overlap over one extremity is needed for recovering the successor constraints on trains for the colimit. We obtain similar links between global and local properties also with a cover consisting of one-train systems. However, then the colimit of the system defined by such a cover is different of the system S ; we would obtain a link between the safety of the systems consisting of one train only and the safety of a system in which all trains are on independent tracks.

10.2 Example 2: Lifeness

We adapt the example in the previous section and give an example of lifeness property which can be expressed by means of a cartesian theory, and thus can be checked modularly. Assume that the constraints Γ'_j on for system S_j consist of Γ_j (defined as Γ_k^l in Example 5) and the constraint $(\bigwedge_{i \in I_j} \text{Mode}_i = 0) \vee (\prod_{i \in I_j} \text{Mode}_i > 0)$. As in Theorem 52 we can prove that this defines a subsheaf St'_j of St ; the following constraints define subsheaves of St' with properties similar to those of SafeState :

- $\Gamma_{su}^j = \Gamma'_j \cup \Gamma_s^j \cup \{\text{Mode}_i = 0 \mid i \in I_j\}$ defines a sheaf SafeStateUpdate ;
- $\Gamma_{\text{CanMove}}^j = \Gamma'_j \cup \{\text{Mode}_i > 0 \mid i \in I_j\}$ defines a sheaf CanMove ;
- $\Gamma_{\text{CannotMove}}^j = \Gamma'_j \cup \{\text{Mode}_i = 0 \mid i \in I_j\}$ defines a sheaf CannotMove .

For $S_i \in \text{InSys}$ let $\text{Minimal}(S_i) = \{(h, j) \mid s(h, j) \in \text{CanMove}(S_i) \text{ and } \forall k(s(h, k) \in \text{CanMove}(S_i) \rightarrow k \geq j)\}$, characterizing the minimal moment in time j w.r.t. a behavior h at which all trains in system S_i can move. These definitions can be used to define a subsheaf $\text{MinimalCanMove} \subseteq \mathbb{B}_{\mathbb{N}} \times \mathbb{N}$ with properties similar to those of $\text{St}, \text{Pa}, \text{Tr}, \text{B}$. A form of lifeness can be expressed by the following cartesian axioms:

$$\begin{aligned} \forall h : \mathbf{b} \quad & (\text{SafeStateUpdate}(s(h, 0)) \rightarrow \exists j : \mathbf{t} \text{ MinimalCanMove}(h, j)) \\ \forall h : \mathbf{b}, \forall i : \mathbf{t} \quad & (\text{MinimalCanMove}(h, i) \rightarrow \text{CanMove}(s(h, i))) \\ \forall h : \mathbf{b}, \forall i, k : \mathbf{t} \quad & (\text{MinimalCanMove}(h, i) \wedge \text{CanMove}(s(h, k)) \rightarrow i \leq k) \end{aligned}$$

(where the existential quantified variable in the first axiom is provably unique modulo the second and third axiom), and can thus be checked modularly.

11 Conclusion

We showed that a family InSys of interacting systems closed under pullbacks can be endowed with a topology which models the way these systems interact.

States, parallel actions, transitions, and behavior can be described as sheaves on this topological space. We then used geometric logic to determine which kind of properties of systems in `InSys` are preserved when interconnecting these systems. The main advantage of our approach is that it enables us to verify properties of complex systems in a modular way. We illustrated the ideas by means of a running example, involving systems of trains controlled by interacting controllers. In future work we plan to look at other applications, including geographically distributed systems, controlled by geographically fixed controllers, whose domains overlap.

We think that there should exist relationships between the approach described in this paper and other new approaches to the study of concurrency such as, for instance, higher dimensional automata (cf. [14,15]) or approaches based on methods from geometry and algebraic topology in particular homotopic methods (cf. [7]). Links between algebraic topology and concurrency as well as links with higher dimensional automata have been studied e.g. by Gaucher, Goubault, Fajstrup, and Raussen (cf. e.g. [5,4]). We would like to compare our approach with the methods mentioned above. Using homological and especially homotopic methods seems to be the next natural step after the sheaf semantics given in this paper.

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